Euclid eWorkshop # 1

Logarithms and Exponents
**TOOLKIT**

If $a$, $b$, $x$, and $y$ are real numbers and $n$ is a nonzero integer then the rules for exponents are:

\[ a^n = \sqrt[n]{a} \quad a^0 = 1 \text{ if } a \neq 0 \quad a^{-x} = \frac{1}{a^x} \text{ if } a \neq 0 \]

\[ a^x a^y = a^{x+y} \quad \frac{a^x}{a^y} = a^{x-y} \text{ if } a \neq 0 \quad (a^x)^y = a^{xy} \]

\[ a^x \cdot b^x = (ab)^x \quad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x \text{ if } b \neq 0 \]

Also, $0^0$ is not defined if it is encountered using any of the above formulae.

If $a$, $x$, and $y$ are nonzero real numbers then:

\[ \log_a (xy) = \log_a x + \log_a y \quad \log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \]

\[ \log_a (x^y) = y \log_a x \quad \log_a (a^x) = a^{\log_a x} = x \quad \log_a 1 = 0 \]

\[ \log_a x = \frac{1}{\log_x a} \quad \frac{\log_a x}{\log_a y} = \log_y x \]

If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$. You should be able to sketch the graphs of both these functions. The graphs are shown for $a = 2$ below.

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![Graph of $a^x$ and $\log_a x$](image-url)
SAMPLE PROBLEMS

1. Calculate the ratio \( \frac{x}{y} \) if \( 2 \log_5(x - 3y) = \log_5(2x) + \log_5(2y) \)

   **Solution**

   First we note that if the 3 logarithmic terms are to be defined in the original equation then their arguments must be positive. So \( x > 0, y > 0, \) and \( x > 3y. \) Now

   \[
   2 \log_5(x - 3y) = \log_5(2x) + \log_5(2y) \\
   \log_5(x - 3y)^2 = \log_5(4xy).
   \]

   Now since the log function takes on each value its range only once, this implies

   \[
   (x - 3y)^2 = 4xy \\
   x^2 - 6xy + 9y^2 = 4xy \\
   x^2 - 10xy + 9y^2 = 0 \\
   (x - y)(x - 9y) = 0
   \]

   So \( \frac{x}{y} = 1 \) or 9. But \( \frac{x}{y} > 3 \) from our restrictions so that \( \frac{x}{y} = 9. \)

2. If \( m \) and \( k \) are integers, find all solutions to the equation \( 9(7^k + 7^{k+2}) = 5^{m+3} + 5^m. \)

   **Solution**

   We factor both sides of this equation to arrive at:

   \[
   9(1 + 7^2)7^k = 5^m(1 + 5^3) \\
   3^2 \cdot 2 \cdot 5^2 \cdot 7^k = 5^m \cdot 2 \cdot 3^2 \cdot 7
   \]

   Now since both sides of this equation are integers and have unique factorizations it follows that \( m = 2 \) and \( k = 1 \) is the only solution.

3. Determine the points of intersection of the curves \( y = \log_{10}(x - 2) \) and \( y = 1 - \log_{10}(x + 1). \)

   **Solution**

   Again the arguments of the log functions, \( x - 2 \) and \( x + 1 \) must be positive which implies that \( x > 2. \) Now
\[ \log_{10}(x - 2) = 1 - \log_{10}(x + 1) \]
\[ \log_{10}(x - 2) + \log_{10}(x + 1) = 1 \]
\[ \log_{10}([x - 2](x + 1)) = 1 \]
\[ (x - 2)(x + 1) = 10 \]
\[ x^2 - x - 2 = 10 \]
\[ x^2 - x - 12 = 0 \]
\[ (x - 4)(x + 3) = 0 \]

So \( x = 4 \) or \(-3\), but \( x > 2 \) from our restrictions so \( x = 4 \). The point of intersection is \((4, \log_{10} 2)\) or \((4, 1 - \log_{10} 5)\). Since \( \log_{10} 2 + \log_{10} 5 = 1 \), these are equivalent answers.

4. Solve for \( x \) if \( \log_2(9 - 2^x) = 3 - x \).

**Solution**

Once again the argument of the log must be positive, implying that \( 9 > 2^x \).

\[ \log_2(9 - 2^x) = 3 - x \]
\[ (9 - 2^x) = 2^{3-x} = \frac{8}{2^x} \]

Substituting \( y = 2^x \) we have:

\[ 9 - y = \frac{8}{y} \]
\[ y^2 - 9y + 8 = 0 \]

Thus, \( y = 1 \) or \( y = 8 \). Substituting back in, we see that \( x = 0 \) or \( x = 3 \). Both of these satisfy the restriction.

5. The graph of \( y = m^x \) passes through the points \((2, 5)\) and \((5, n)\). What is the value of \( mn \)?

**Solution**

From the given information, \( m^2 = 5 \) and \( n = m^5 \). Thus

\[ m = \pm \sqrt[5]{5} \]
\[ n = (\pm \sqrt[5]{5})^5 \]
\[ mn = (\sqrt[5]{5})^6 = 125. \]
1. Determine $x$ such that $\log_x 2 + \log_x 4 + \log_x 8 = 1$.

2. Determine the values of $x$ such that $12^{2x+1} = 2^{3x+7} \cdot 3^{3x-4}$.

3. What is the sum of the following series?
   \[ \log_{10} \frac{3}{2} + \log_{10} \frac{4}{3} + \log_{10} \frac{5}{4} + \cdots + \log_{10} \frac{200}{199}. \]

4. If $x^3 y^5 = 2^{11} \cdot 3^{13}$ and $\frac{x}{y^2} = \frac{1}{27}$, solve for $x$ and $y$.

5. If $\log_a 3 = k$ then express $\log_a 18$ in terms of $k$.

6. Solve the equations for the point of intersection of the graphs of $y = \log_2 (2x)$ and $y = \log_4 x$.

7. The points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on the graph of $y = \log_a x$. Through the midpoint of $AB$ a horizontal line is drawn to cut the curve at $C(x_3, y_3)$. Show that $(x_3)^2 = x_1 x_2$.

8. The function $y = ax^r$ passes through the points $(2, 1)$ and $(32, 4)$. Calculate the value of $r$.

9. If $2^{x+3} + 2^x = 3^{y+2} - 3^y$ and $x$ and $y$ are integers, determine the values of $x$ and $y$.

10. If $f(x) = 2^{4x-2}$, calculate, in simplest form, $f(x) \cdot f(1-x)$.

11. Find all values of $x$ such that $\log_5 (x - 2) + \log_5 (x - 6) = 2$.

12. Prove that $a$, $b$, and $c$ are 3 numbers that form a geometric sequence if and only if $\log_a a$, $\log_b b$ and $\log_c c$ form an arithmetic sequence.