Solutions

1. Subtract one equation from the other and factor the resulting expression.

\[ xy + y - 8 - 8x = 0 \]
\[ x(y - 8) + y - 8 = 0 \]
\[ (x + 1)(y - 8) = 0 \]

There are solutions when \( x = -1 \) and when \( y = 8 \). If \( x = -1 \) then \( y = -9 \). If \( y = 8 \) then \( x = 4 \pm 2\sqrt{2} \). The solutions are \((-1, -9)\) and \((4 \pm 2\sqrt{2}, 8)\).

2. We are asked for the \( x \) value of the midpoint of zeros, which is the \( x \) value of the vertex. The equation is written in vertex form already, having an \( x \) value of 1.

Alternately Solution: Find the intercepts:

\[ (x - 1)^2 - 4 = 0 \]
\[ (x - 1)^2 = 4 \]
\[ x = 1 \pm 2 \]

Thus \( x = 3 \) or \(-1\). Thus \( a = \frac{-1 + 3}{2} = 1 \).

3. (a) Consider \( a = 0 \) and \( a = 1 \), and find the intersection point of the resulting equations, \( y = x^2 \) and \( y = x^2 + 2x + 1 \). Then \( 0 = 2x + 1 \) and the intersection point is \((-\frac{1}{2}, \frac{1}{4})\). Now substitute this point into the general equation to show that this point is on all the parabolas, since

\[ y = x^2 + 2ax + a \]
\[ = \frac{1}{4} + 2a \cdot -\frac{1}{2} + a \]
\[ = \frac{1}{4} \]

(b) Now \( y = x^2 + 2ax + a = (x + a)^2 + a - a^2 \) so the vertex is at \((-a, a - a^2)\). If we represent the coordinates of the vertex by \((p, q)\) we have \( p = -a \) and \( q = a - a^2 \) or \( q = -p^2 - p \), the required parabola.

4. (a)
(b) From the graph \( x \geq 0 \).

5. Factoring both equations we arrive at:

\[
\begin{align*}
    p(1 + r + r^2) &= 26 \quad (1) \\
    p^2 r(1 + r + r^2) &= 156 \quad (2)
\end{align*}
\]

Dividing (2) by (1) gives \( pr = 6 \). Substituting this relation back into (1) we get

\[
\begin{align*}
    6 + 6r &= 26 \\
    6 - 20r + 6r^2 &= 0 \\
    3r^2 - 10r + 3 &= 0 \\
    (3r - 1)(r - 3) &= 0
\end{align*}
\]

Hence \( (r, p) = (3, 2) \) or \( (\frac{1}{3}, 18) \).

6. We assume, on the contrary, that the coefficients are in geometric sequence. Then \( \frac{b}{a} = \frac{c}{b} \) or \( b^2 = ac \). But now the discriminant \( b^2 - 4ac = -3b^2 < 0 \) so that the roots are not real. Thus we have a contradiction of the condition set out in the statement of the problem and our assumption is false.

7. Let \( r \) and \( s \) be the integer roots. The equation can be written as

\[
a(x - r)(x - s) = a(x^2 - (r + s)x + rs)
\]

\[
= ax^2 - a(r + s)x + ars
\]

\[
= ax^2 + bx + c
\]

with \( b = -a(r + s) \) and \( c = ars \). Since \( a, b, c \) are in arithmetic sequence, we have

\[
\begin{align*}
    c - b &= b - a \\
    a + c - 2b &= 0 \\
    a + ars + 2a(r + s) &= 0 \\
    1 + rs + 2(r + s) &= 0 \quad \text{we can divide by } a \text{ since } a \neq 0 \\
    (r + 2)(s + 2) &= 3
\end{align*}
\]

Since there are only 2 integer factorings of 3 we have \( \{r, s\} = \{1, -1\} \) or \( \{-3, -5\} \).

8. Solution 1

Multiplying out and collecting terms results in \( x^4 - 6x^3 + 8x^2 + 2x - 1 = 0 \). We look for a factoring with integer coefficients, using the fact that the first and last coefficients are 1. So

\[
x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 + ax + 1)(x^2 + bx - 1)
\]

where \( a \) and \( b \) are undetermined coefficients. However multiplication now gives \( a + b = -6 \) and \( -a + b = 2 \) and \( ab = 8 \). Since all 3 equations are satisfied by \( a = -4 \) and \( b = -2 \), we have factored the original expression as

\[
x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)
\]

Factoring these two quadratics gives roots of \( x = 2 \pm \sqrt{3} \) and \( x = 1 \pm \sqrt{2} \).
Solution 2
We observe that the original equation is of the form \( f(f(x)) = x \) where \( f(x) = x^2 - 3x + 1 \). Now if we can find \( x \) such that \( f(x) = x \) then \( f(f(x)) = x \). So we solve \( f(x) = x^2 - 3x + 1 = x \) which gives the first factor \( x^2 - 4x + 1 \) above. With polynomial division, we can then determine that
\[
x^4 - 6x^3 + 8x^2 + 2x - 1 = (x^2 - 4x + 1)(x^2 - 2x - 1)
\]
and continue as in Solution 1.

9. The vertex has \( x = 2 \) and \( y = -16 \) so \( A = (2, -16) \). When \( y = 0 \) we get intercepts at \(-2\) and \(6\). The larger value is \(6\), so \( B = (6, 0) \). Therefore we want the line through \((2, -16)\) and \((6, 0)\) which is
\[
4x - y - 24 = 0.
\]

10. Solution 1
Multiplying gives
\[
x^2 - (b + c)x + bc = a^2 - (b + c)a + bc
\]
\[
0 = x^2 - (b + c)x + a(b + c - a)
\]
\[
x = \frac{b + c \pm \sqrt{(b + c)^2 - 4a(b + c - a)}}{2}
\]
\[
= \frac{b + c \pm \sqrt{(b + c - 2a)^2}}{2}
\]
\[
= a \text{ OR } b + c - a
\]

Solution 2 Observe that \( x = a \) is one solution. Rearrange as above to get \( x^2 - (b + c)x + a(b + c - a) = 0 \).
Using the sum/product of roots, the other solution is \( x = b + c - a \).

11. Since \( x = -2 \) is a solution of \( x^3 - 7x - 6 \), thus \( x + 2 \) is a factor. Factor as
\[
x^3 - 7x - 6 = (x + 2)(x^2 - 2x - 3) = (x + 2)(x + 1)(x - 3)
\]
so the roots are \(-2, -1\) and \(3\).

12. Let the roots be \( r \) and \( s \). By the sum and product rule,
\[
r + s = \frac{-4(a - 2)}{4} = 2 - a
\]
\[
rs = \frac{-8a^2 + 14a + 31}{4} = -2a^2 + \frac{7}{2}a + \frac{31}{4}
\]
Then
\[
r^2 + s^2 = (r + s)^2 - 2rs
\]
\[
= (2 - a)^2 - 2(-2a^2 + \frac{7}{2}a + \frac{31}{4})
\]
\[
= 4 - 4a + a^2 + 4a^2 - 7a - \frac{31}{2}
\]
\[
= 5a^2 - 11a - \frac{23}{2}.
\]
It appears that the minimum value should be at the vertex of the parabola \( f(a) = 5a^2 - 11a - \frac{23}{2} \), that is at \( a = \frac{11}{10} \) (found by completing the square). But we have ignored the condition that the roots are real. The discriminant of the original equation is

\[
B^2 - 4AC = [4(a - 2)]^2 - 4(4)(-8a^2 + 14a + 31) = 16(a^2 - 4a + 4) + 128a^2 - 224a - 496 = 144a^2 - 288a - 432 = 144(a - 3)(a + 1).
\]

Thus we have real roots only when \( a \geq 3 \) or \( a \leq -1 \). Therefore \( a = \frac{11}{10} \) cannot be our final answer, since the roots are not real for this value. However \( f(a) = 5a^2 - 11a - \frac{23}{2} \) is a parabola opening up and is symmetrical about its axis of symmetry \( a = \frac{11}{10} \). So we move to the nearest value of \( a \) to the axis of symmetry that gives real roots, which is \( a = 3 \).

13. Let \( g(2) = k \). Since \( f \) and \( g \) are inverse functions, thus \( f(k) = 2 \). We need to solve

\[
\frac{3k - 7}{k + 1} = 2
\]

\[
3k - 7 = 2(k + 1)
\]

\[
k = 9
\]

Thus \( g(2) = 9 \).

14. Write

\[
y = -2x^2 - 4ax + k
\]

\[
= -2(x^2 + 2ax + \frac{k}{2})
\]

\[
= -2(x + a)^2 + k + 2a^2
\]

The vertex is at \((-a,k + 2a^2)\) or \((-2,7)\) and we can solve for \( a = 2 \) and \( k = -1 \).

15. Using sum and product of roots we have the 4 equations:

\[
a + b = -c
da = d
\]

\[
c + d = -a\cd = b.
\]

Therefore

\[
-(c + d) + cd = -c
\]

\[
= cd - d = 0
\]

\[
d(c - 1) = 0
\]

But none of \( a, b, c \) or \( d \) are zero, so \( c = 1 \). Then we get \( d = b, a = 1 \) and \( d = b = -2 \). Thus \( a + b + c + d = -2 \).
16. The most common way to do this problem uses calculus. However we make the substitution $z = x - 4$. To get $y$ in terms of $z$, try

$$y = x^2 - 2x - 3$$
$$= (x - 4)^2 + 6x - 19$$
$$= (x - 4)^2 + 6(x - 4) + 5$$
$$= z^2 + 6z + 5$$

The value we want to minimize is then

$$\frac{y - 4}{(x - 4)^2} = \frac{z^2 + 6z + 1}{z^2} = 1 + \frac{6}{z} + \frac{1}{z^2}.$$ If we now let $u = \frac{1}{z}$, we have the up-opening parabola $1 + 6u + u^2$ which has its minimum at $u = -3$ with minimum value of $-8$. Note that since $x$ can assume any real value except 4, $z$ and $u$ will assume all real values except zero. Thus the minimum value of this expression is $-8$. 