Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 3

Solutions

SOLUTIONS

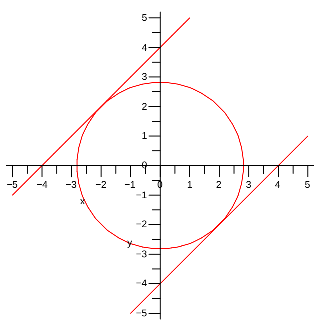
1. Using the line segment from $O(0,0)$ to $C(9,0)$ as the base and noting the height is 4, the area of triangle OCD is 18. We let the vertical line be $x = k$. The line from $O(0,0)$ to $D(8,4)$ is $y = \frac{1}{2}x$ and this intersects the vertical line at $K(k, 1/2k)$. Let $L = (k,0)$ be the x intercept of the vertical line. The area of triangle OKL must be $\frac{1}{4}k^2 = 9$ and so the vertical line required is $x = 6$.

2. There are several ways to do this question; we proceed using analytic geometry. If the line is tangent to the circle, then the distance from the centre (0,0) to the line $y = x + c$ (or $(x - y + c = 0)$) equals the radius of the circle, $2\sqrt{2}$. Using the formula for distance from a point to a line in the toolkit,

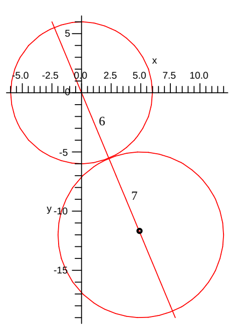
$$D = \frac{|\ Ax_0 + By_0 + C\ |}{\sqrt{A^2 + B^2}}$$

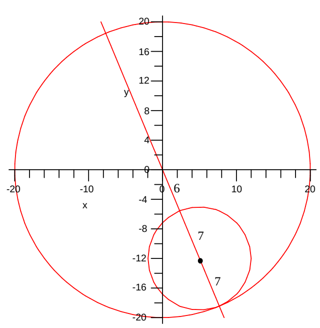
$$2\sqrt{2} = \frac{|\ c\ |}{\sqrt{2}}$$

Therefore we have $c = \pm 4$.



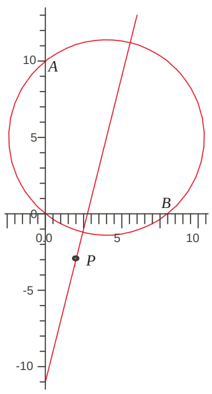
3. There are two circles, the first with centre (0,0) and radius k, and the second with centre $(5,-12)$ and radius 7. The distance between the centres can be calculated to be $\sqrt{(-5)^2 + (12)^2} = 13$. Now if the two circles intersect only once, they can be either externally or internally tangent. If they are externally tangent, $k + 7 = 13$ and $k = 6$. If they are internally tangent, $|\ k - 7\ | = 13$ and $k = 20$ or -6. But the radius must be positive so $k = 6$ or 20.



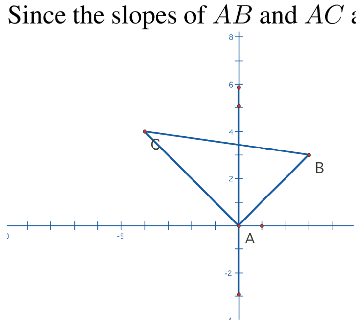


4. **Solution 1:** All lines that cut a circle in half pass through the centre. Now the perpendicular bisector of any chord passes through the centre. If we consider the vertical chord from (0,0) to (0,10), the perpendicular bisector is the horizontal line $y = 5$. Similarly if we consider the horizontal chord from (0,0) to (8,0), the perpendicular bisector is the vertical line $x = 4$. Therefore the centre is (4,5). We require the y intercept of the line through (4,5) and $P(2, -3)$. This line is $y = 4x - 11$ and the intercept is -11.

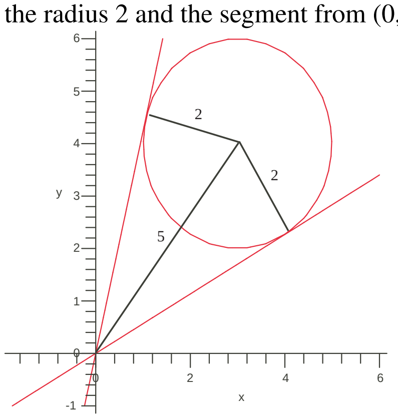
   Solution 2: Observe that $\triangle AOB$ is right-angled at O, thus AB is the diameter of the circle, and its midpoint $(4, 5)$ is the centre of the circle. As in solution 1, we require the y intercept of the line through (4,5) and $P(2, -3)$. This line is $y = 4x - 11$ and the intercept is -11.



5. Since the slopes of AB and AC are 1 and -1 respectively, the required line is vertical and its equation is $x = 0$.



6. The tangent is perpendicular to the radius at the point of tangency. The two known sides of the right triangle are the radius 2 and the segment from (0,0) to (3,4) which has length 5. Thus the tangents are of length $\sqrt{21}$.



   Since the tangent passes through the origin, let its equation be $y = mx$. We are interested in values of m for which the line $y = mx$ intersects the circle only once. Substituting into the equation of the circle we get

$$\begin{aligned}(x-3)^2 + (mx-4)^2 &= 4\\ x^2 - 6x + 9 + m^2x^2 - 8mx + 16 &= 4\\ (1+m^2)x^2 - (6+8m)x + 21 &= 0\end{aligned}$$

Now this quadratic will have one solution when its discriminant its zero; we are looking for values of m that give a discriminant of 0.

$$\begin{aligned}D = (6+8m)^2 - 4\cdot 21\cdot(1+m^2) &= 0\\ 36 + 96m + 64m^2 - 84 - 84m^2 &= 0\\ -20m^2 + 96m - 48 &= 0\\ m &= \frac{12 \pm 2\sqrt{21}}{5}\end{aligned}$$

7. The required set of points is the line that is the perpendicular bisector of the line segment CD. Since CD has a slope $\frac{-1}{2}$ and a midpoint $M = \left(3, \frac{3}{2}\right)$, the required line passes through M and has slope 2. The resulting equation is $4x - 2y - 9 = 0$.

8. We present the solution that uses analytic geometry most directly. Let the co-ordinates of the points be $K(0{,}0)$, $W(x{,}y)$, $A(a{,}b)$ and $D(d{,}0)$. Therefore the co-ordinates of M and N are $M\left(\frac{x}{2}, \frac{y}{2}\right)$ and $N\left(\frac{a+d}{2}, \frac{b}{2}\right)$. Now we are given that $2MN = AW + DK$. Therefore

$$\begin{aligned}2\sqrt{\left(\frac{a+d-x}{2}\right)^2 + \left(\frac{b-y}{2}\right)^2} &= d + \sqrt{(a-x)^2 + (b-y)^2}\\ (a+d-x)^2 + (b-y)^2 &= d^2 + (a-x)^2 + (b-y)^2 + 2d\sqrt{(a-x)^2+(b-y)^2} \quad \text{after squaring}\\ 2d(a-x) &= 2d\sqrt{(a-x)^2 + (b-y)^2}\\ (a-x)^2 &= (a-x)^2 + (b-y)^2 \quad \text{since } d \neq 0, \text{ squaring both sides}\\ (b-y)^2 &= 0\end{aligned}$$

This result gives $b = y$ and implies that the slope of $AW = 0$ and hence that AW is parallel to KD.

9. If $A(a{,}c)$ and $B(b{,}d)$ then $4a + 3c - 48 = 0$ and $b + 3d + 10 = 0$ since these points lie on each of the 2 lines. Moreover, since (4,2) is the midpoint, we know $\frac{a+b}{2} = 4$ and $\frac{c+d}{2} = 2$. Thus $b = 8 - a$ and $d = 4 - c$, which together with the linear equations above, give $A(6{,}8)$ and $B(2{,}-4)$.