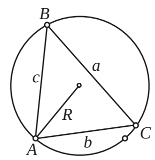
Centre for Education
in Mathematics and Computing

Euclid eWorkshop # 4

Trigonometry

TOOLKIT

Name	Formula
1. Sine Law	$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ where R is the radius of the circumcircle 
2. Cosine Law	$a^2 = b^2 + c^2 - 2bc\cos A$ $b^2 = a^2 + c^2 - 2ac\cos B$ $c^2 = b^2 + a^2 - 2ab\cos C$
3. Area relations	The area of triangle $ABC = \|\triangle ABC\| = \dfrac{1}{2}ab\sin C = \dfrac{1}{2}bc\sin A = \dfrac{1}{2}ac\sin B$.
4. Area of an equilateral triangle	The area of an equilateral triangle of side length s is $\dfrac{\sqrt{3}s^2}{4}$.
5. Heron's formula	$\|\triangle ABC\| = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$.
6. Trigonometric Identities	$\tan\theta = \dfrac{1}{\cot\theta} = \dfrac{\sin\theta}{\cos\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$
7. Related Angle Identities	$\sin(180^\circ - \theta) = \sin\theta$ $\cos(180^\circ - \theta) = -\cos\theta$
8. Draw graphs of	$y = A\sin(kx+d)$ $y = A\cos(kx+d)$ $y = \tan x$ For the sine and cosine graphs that the amplitude is A, the period is $\dfrac{2\pi}{k}$, and the phase shift is $\dfrac{-d}{k}$.

SAMPLE PROBLEMS

1. Calculate the roots of $2\sin^3 x - 5\sin^2 x + 2\sin x = 0$ in the range $0 \le x \le 2\pi$.

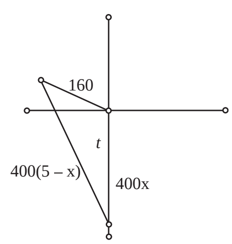
Solution

We factor this equation
$2\sin^3 x - 5\sin^2 x + 2\sin x = 0$
$\sin x(2\sin^2 x - 5\sin x + 2) = 0$
$\sin x(2\sin x - 1)(\sin x - 2) = 0$
So $\sin x = 0$, $\frac{1}{2}$ or 2. But $|\sin x| \le 1$. So $x = 0$, π, 2π, $\frac{\pi}{6}$, $\frac{5\pi}{6}$.

2. An airplane leaves an aircraft carrier and flies due south at 400 km/hr. The carrier proceeds at a heading of $60°$ west of north at 32 km/hr. If the plane has 5 hours of fuel, what is the maximum distance south the plane can travel so that the fuel remaining will allow a safe return to the carrier at 400 km/hr?

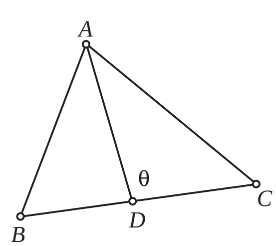
Solution

The first step in solving this problem is to draw a diagram (as shown). If we let x be the number of hours that the plane flies south then the distance that the plane flies south is $400x$. The plane then flies a distance $400(5-x)$. in the remaining time while the total distance the carrier flies is $5(32)$. Using these distances the cosine law states: $(400(5-x))^2 = 160^2 + (400x)^2 - 2 \cdot 160 \cdot 400x \cdot \cos 120°$.



If we replace the $\cos 120°$ with $-\frac{1}{2}$ and divide through by 6400 we get:
$25(5-x)^2 = 4 + 25x^2 + 10x$ or $x = \frac{621}{260}$. Thus, the maximum distance is approximately 955.4 km..

3. In triangle ABC the point D is on BC such that AD bisects $\angle A$. Show that $\frac{AB}{BD} = \frac{AC}{CD}$.

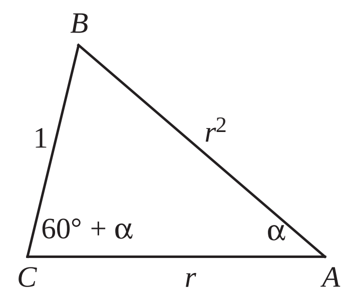


Solution

The ratios suggest use of the sine law. We call $\angle ADC = \theta$. So if we use the sine law in the triangles ADC and ADB we have $\dfrac{\sin\dfrac{A}{2}}{\sin\theta} = \dfrac{CD}{AC}$ and $\dfrac{\sin\dfrac{A}{2}}{\sin(180^\circ - \theta)} = \dfrac{BD}{AB}$. But $\sin\theta = \sin(180^\circ - \theta)$ and so $\dfrac{AB}{BD} = \dfrac{AC}{CD}$.

This result is known as the angle bisector theorem.

4. For the given triangle ABC, $\angle C = \angle A + 60^\circ$. If $BC = 1$, $AC = r$ and $AB = r^2$, where $r > 1$, prove that $r < \sqrt{2}$.



Solution

This problem was the last problem in the 1996 Euclid, an indication that it is quite difficult. Many difficult problems involve the use of several tools, and this will involve the sine law, the cosine law and an inequality.

We represent the angles of the triangle as: $\angle A = \alpha$; $\angle C = \alpha + 60^\circ$; $\angle B = 120^\circ - 2\alpha$. So the sine law states

$$\begin{aligned}\frac{r^2}{1} &= \frac{\sin(\alpha + 60^\circ)}{\sin\alpha} \\ &= \frac{\sin\alpha\cos 60^\circ + \sin 60^\circ\cos\alpha}{\sin\alpha} \\ &= \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2}\end{aligned}$$

Since all 3 angles in the triangle are positive, we can see that $0 < \alpha < 60^\circ$. In this range the tangent function is increasing, and its reciprocal, the cotangent function, is decreasing.

The cosine law gives

$$r^2 = 1 + r^4 - 2r^2\cos(120^\circ - 2\alpha) \quad (1).$$

But

$$(r^2 - 1)^2 \geq 0 \text{ or } r^4 + 1 \geq 2r^2 \quad (2).$$

Substituting (2) into (1) gives $r^2 \geq 2r^2 - 2r^2\cos(120^\circ - 2\alpha)$ which implies $\cos(120^\circ - 2\alpha) \geq \dfrac{1}{2}$. Thus $\alpha \geq 30^\circ$ and

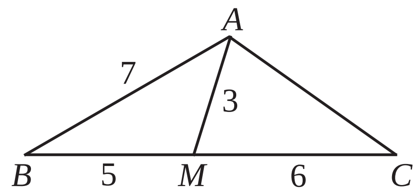
$r^2 = \dfrac{\sqrt{3}}{2}\cot\alpha + \dfrac{1}{2} \leq \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{1} + \dfrac{1}{2} \leq 2$ and we are done.

PROBLEM SET

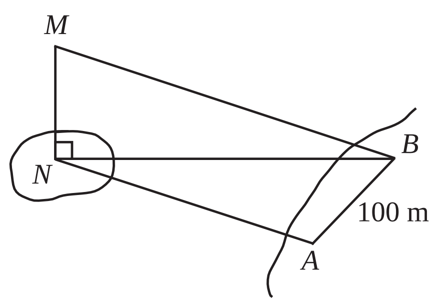
1. (a) If $2\sin(2\theta)+1=0$, find the smallest positive value of θ (in degrees).

   (b) For $-\pi \le \theta \le \pi$, find all solutions to the equation $2(\sin^2\theta - \cos^2\theta) = 8\sin\theta - 5$.

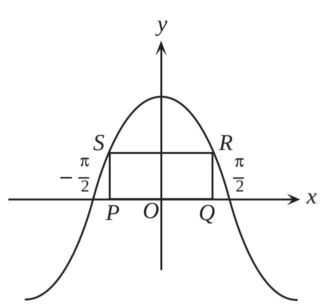
2. In $\triangle ABC$, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



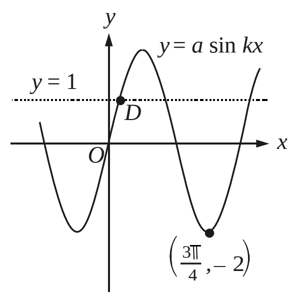
3. In determining the height, MN, of a tower on an island, two points A and B, 100 m apart, are chosen on the same horizontal plane as N. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



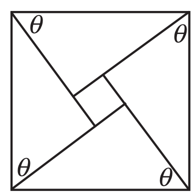
4. A rectangle $PQRS$ has side PQ on the x-axis and touches the graph of $y = k\cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k?



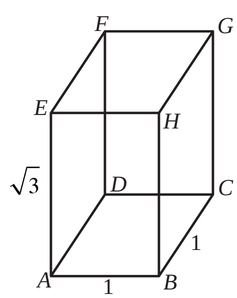
5. The graph of the equation $y = a \sin kx$ is shown in the diagram, and the point $\left(\dfrac{3\pi}{4}, -2\right)$ is the minimum point indicated. The line $y = 1$ intersects the graph at point D. What are the coordinates of D?



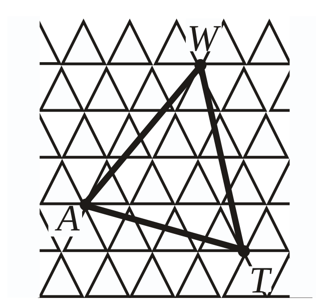
6. A square with an area of 9 cm^2 is surrounded by four congruent triangles, forming a larger square with an area of 89 cm^2. If each of the triangles has an angle θ as shown, find the value of $\tan\theta$.



7. A rectangular box has a square base of length 1 cm, and height $\sqrt{3}$ cm as shown in the diagram. What is the cosine of angle FAC?



8. In the grid, each small equilateral triangle has side length 1. If the vertices of $\triangle WAT$ are themselves vertices of small equilateral triangles, what is the area of $\triangle WAT$?



9. In triangle ABC, $AB = 8$, and $\angle CAB = 60°$. Sides BC and AC have integer lengths a and b, respectively. Find all possible values of a and b.