Euclid eWorkshop # 4

Solutions
SOLUTIONS

1. (a) Here \( \sin 2\theta = -\frac{1}{2} \). Thus \( 2\theta = 210^\circ \) and \( \theta = 105^\circ \).

(b) 
\[
\cos^2(\theta) = 1 - \sin^2(\theta) \\
2(2\sin^2(\theta) - 1) = 8\sin\theta - 5 \\
4\sin^2(\theta) - 8\sin\theta + 3 = 0 \\
(2\sin\theta - 1)(2\sin\theta - 3) = 0 \\
\sin\theta = \frac{1}{2}, \frac{3}{2} \\
\text{but } |\sin\theta| \leq 1 \\
\text{So } \theta = \frac{\pi}{6}, \frac{5\pi}{6}
\]

2. Let \( \theta = \angle AMC \). Using the cosine law in \( \triangle ABM \) gives
\[
49 = 9 + 25 - 30\cos(180^\circ - \theta) \\
15 = -30\cos(180^\circ - \theta) \\
\cos(180^\circ - \theta) = -\frac{1}{2} \\
\cos(\theta) = -\cos(180^\circ - \theta) \\
= \frac{1}{2}
\]
Using the cosine law in \( \triangle AMC \) gives
\[
AC^2 = 9 + 36 - 36\cos(\theta) \\
= 27 \\
AC = 3\sqrt{3}.
\]

3. Use the sine law: \( \frac{BN}{\sin(108^\circ)} = \frac{100}{\sin(25^\circ)} \). But \( MN = \tan 32^\circ \). So \( MN = 100 \cdot \frac{\sin 108^\circ}{\sin 25^\circ} \cdot \tan 32^\circ \approx 141 \text{ m.} \)

4. Since the area of the rectangle is \( \frac{5\pi}{3} \), its height is 5. Since the cosine graph is symmetrical about the \( y \)-axis,
\[
PO = OQ = \frac{\pi}{6}. \text{ But } \cos \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{2}. \text{ So } k = \frac{10\sqrt{3}}{3}.
\]

5. Since the minimum point has a \( y \)-coordinate of -2, the amplitude is \( a = 2 \). Also since the minimum occurs at \( x = \frac{3\pi}{2} \) (rather than \( \frac{3\pi}{2} \) where it is found for \( \sin(x) \)), \( k = 2 \). Therefore \( \sin(2x) = \frac{1}{2} \) and \( x = \frac{\pi}{12} \). Thus \( D = \left( \frac{\pi}{12}, 1 \right) \).

6. We let the side of the triangles opposite \( \theta \) be \( a \) and side leg adjacent to \( \theta \) be \( b \). Then \( \tan \theta = \frac{a}{b} \), \( a - b = 3 \) and
\[
4 \left( \frac{1}{2} ab \right) = 89 - 9 = 80 \text{ and } b = \frac{40}{a}. \text{ Thus } a - \frac{40}{a} = 3 \text{ or } a^2 - 3a - 40 = 0 \text{ which gives } a = 8 \text{ or } -5. \text{ Now
since $a$ is positive, $a = 8$ and $b = 5$ and $\tan \theta = \frac{8}{5}$.

7. Using Pythagorean theorem, we find that $FA = 2$, $AC = \sqrt{2}$ and $FC = 2$. The cosine law in $\triangle FAC$ gives

$$FC^2 = FA^2 + AC^2 - 2 \cdot FA \cdot AC \cdot \cos(\angle FAC)$$
$$4 = 4 + 2 - 2 \cdot 2 \cdot \sqrt{2} \cos(\angle FAC)$$
$$\cos(\angle FAC) = \frac{1}{2\sqrt{2}}.$$  

8. Using item 4 from the toolkit, the height of each small equilateral triangle is $\frac{\sqrt{3}}{2}$. Let $O$ be the vertex immediately to the right of $W$, and consider right-angled triangle $WOT$. Now $OT$ is the height of four of the small triangles, thus $OT = 2\sqrt{3}$. Also $WO = 1$. By the Pythagorean theorem, we have $WT = \sqrt{1 + 4 \cdot 3} = \sqrt{13}$. Again using item 4 from the toolkit, we have the area of $\triangle WAT = \frac{13\sqrt{3}}{4}$.

9. The cosine law states

$$a^2 = 64 + b^2 - 16b(\cos 60^\circ)$$
$$= b^2 - 8b + 64$$
$$= (b - 4)^2 + 48$$
$$a^2 - (b - 4)^2 = 48$$
$$(a + b - 4)(a - b + 4) = 48$$

But $48 = 24 \cdot 2 = 12 \cdot 4 = 8 \cdot 6$, where we have only considered the even-even factorings of 48 since the 2 brackets on the left side must have the same parity (evenness or oddness). Thus $(a,b) = (13,15), (8,8), (7,5)$ or $(7,3)$. 