Euclid eWorkshop # 5

Sequences and Series
While the vast majority of Euclid questions in this topic area use formulae for arithmetic or geometric sequences, we will also include a few involving summations and different types of sequences.

**TOOLKIT**

**Arithmetic Sequences**

<table>
<thead>
<tr>
<th>Description</th>
<th>Sequences with a common difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>General ( k \text{th} ) term</td>
<td>( t_k = a + (k - 1)d ) where ( a ) is the first term and ( d ) is this common difference</td>
</tr>
<tr>
<td>Sum of ( n ) terms</td>
<td>( S_n = \frac{n}{2}(a + t_n) = \frac{n}{2}(2a + (n - 1)d) )</td>
</tr>
<tr>
<td>Spacing of terms</td>
<td>Because of the equal spacing of terms we have ( t_k + t_l = t_m + t_n ) if and only if ( k + l = m + n )</td>
</tr>
</tbody>
</table>

**Geometric Sequences**

<table>
<thead>
<tr>
<th>Description</th>
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</tr>
</thead>
<tbody>
<tr>
<td>General ( k \text{th} ) term</td>
<td>( t_k = ar^{k-1} ) where ( a ) is the first term and ( r ) is the ratio</td>
</tr>
<tr>
<td>Sum of ( n ) terms</td>
<td>( S_n = \frac{a(1 - r^n)}{1 - r} )</td>
</tr>
<tr>
<td>Spacing of terms</td>
<td>Because of the equal spacing of terms we have ( t_k t_l = t_m t_n ) if and only if ( k + l = m + n )</td>
</tr>
<tr>
<td>Infinite sum</td>
<td>If the ratio ( r ) satisfies the condition (</td>
</tr>
</tbody>
</table>

**Other**

Of course arithmetic and geometric sequences are a small subset of all sequences, even though they are emphasized in high school mathematics. Some extensions that frequently appear on contests often involve:

| First \( n \) integers | \( \sum_{k=1}^{n} k = \frac{n(n + 1)}{2} \) |
| First \( n \) squares | \( \sum_{k=1}^{n} k^2 = \frac{n(n + 1)(2n + 1)}{6} \) |
| First \( n \) cubes | \( \sum_{k=1}^{n} k^3 = \left(\frac{n(n + 1)}{2}\right)^2 \) |
| Telescoping series | If \( t_k = u_k - u_{k-1} \) then \( \sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (u_k - u_{k-1}) = u_n - u_0 \) |
SAMPLE PROBLEMS

1. What is the sum of all multiples of 7 or 11 less than 1000?

Solution

Since we are adding \((7 + 14 + 21 + 28 \ldots 994) + (11 + 22 + 33 + \ldots 990)\), we are adding two arithmetic sequences. However the multiples of 77 are included in both sequences and so must be subtracted (in order to avoid counting them twice) after we add the 2 sequences above. Therefore the required sum is
\[(7 + 14 + 21 + 28 \ldots 994) + (11 + 22 + 33 + \ldots 990) - (77 + 154 + \ldots 924)\].

Now since 994 is the 142nd term in the first sequence we have the sum of the first sequence is \(\frac{142}{2}(7 + 994)\).
Thus the required sum is
\[
\frac{142}{2}(7 + 994) + \frac{90}{2}(11 + 990) - \frac{12}{2}(77 + 924) \\
= (71 + 45 - 6)(1001) \\
= (110)(1001) \\
= 110110
\]

2. A sequence is given such that \(t_1 = 1\) and \(t_{n+1} = t_n + 3n^2 + 3n + 1\). Evaluate \(t_{100}\).

Solution

Since the difference, \(t_n - t_{n-1}\) is not constant, the series is not arithmetic. Now setting \(n = 1\) we find \(t_2 = 1 + 3 + 3 + 1 = 8\). Setting \(n = 2\) we find \(t_3 = 8 + 12 + 6 + 1 = 27\). These facts suggest \(t_n = n^3\) for every \(n\).
To prove that \(t_n = n^3\) is an alternate definition for the same sequence, we first note that \(t_1 = 1 = 1^3\). Further, consider two adjacent terms in the sequence given by the alternate definition, i.e. \(t_n = n^3\) and \(t_{n+1} = (n + 1)^3\). Then the difference between these terms is
\[
t_{n+1} - t_n = (n + 1)^3 - n^3 \\
= (n^3 + 3n^2 + 3n + 1) - n^3 \\
= 3n^2 + 3n + 1 \\
t_{n+1} = t_n + 3n^2 + 3n + 1
\]
which matches the original definition of the sequence. We have proved that the original sequence can be expressed as \(t_n = n^3\), and thus \(t_{100} = 100^3\).
3. If \( a, b, a + b, \) and \( ab \) are positive numbers that form 4 consecutive terms in a geometric sequence, find \( a \).

**Solution**

The ratios of successive terms will be equal since we have a geometric sequence. So

\[
\frac{a}{b} = \frac{b}{a+b} = \frac{a+b}{ab} \quad (*)
\]

Therefore,

\[
a^2 + ab = b^2 \\
b^2 - ab - a^2 = 0 \\
\left( \frac{b}{a} \right)^2 - \left( \frac{b}{a} \right) - 1 = 0 \\
\left( \frac{b}{a} \right) = \frac{1 + \sqrt{5}}{2}
\]

where we have chosen the positive root since \( a \) and \( b \) are positive. Also from (*)

\[
\frac{a}{b} = \frac{a+b}{ab} \\
a^2 = a + b \\
a = 1 + \frac{b}{a} \\
= \frac{3 + \sqrt{5}}{2}.
\]
**Problem Set**

1. In a geometric series, \(t_5 + t_7 = 1500\) and \(t_{11} + t_{13} = 187500\). Find all possible values for the first three terms.

2. Given that \(a, b, c\) are successive terms in an arithmetic sequence calculate \(x\) if
\[(b - c)x^2 + (c - a)x + (a - b) = 0.\]

3. If \(x, 4, y\) are successive terms in an arithmetic sequence and \(x, 3, y\) are successive terms in a geometric sequence, calculate \(\frac{1}{x} + \frac{1}{y}\).

4. Three different numbers, whose product is 125, are 3 consecutive terms in an arithmetic sequence. Find the numbers.

5. The \(k\)th triangular number is given by
\[T_k = 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2} = \frac{k^2 + k}{2}.\] The first few triangular numbers are 1, 3, 6, 10, 15, 21. Find the sum of the first 200 triangular numbers.

6. If the interior angles of a pentagon form an arithmetic sequence and one interior angle is \(90^\circ\), find all possible values of the largest angle in the pentagon.

7. Find the 4 integers \(a, b, c\) and \(d\) that satisfy the following conditions:
   - the sum of \(b\) and \(c\) is 30
   - the sum of \(a\) and \(d\) is 35
   - the numbers \(a < b < c < d\) are in geometric sequence
   - the sum of the squares of the 4 numbers is 1261

8. A sequence \(t_1, t_2, t_3\) is formed by choosing \(t_1\) at random from the set \(\{1, 2, 3\}\), \(t_2\) at random from the set \(\{4, 5, 6\}\), and \(t_3\) at random from the set \(\{7, 8, 9\}\). What is the probability that \(t_1, t_2, t_3\) is an arithmetic sequence?

9. The sum of 25 consecutive integers is 500. Determine the smallest of the 25 integers.


11. The sum of the first \(n\) terms of a sequence is \(S_n = 3^n - 1\), where \(n\) is a positive integer.
   (a) If \(t_n\) represents the \(n\)th term of the sequence, determine \(t_1, t_2, t_3\).
   (b) Prove that \(\frac{t_{n+1}}{t_n}\) is constant for all values of \(n\).

12. How many terms in the arithmetic sequence 7, 14, 21, . . . are between 40 and 28 001?

13. If \(f\) is a function such that \(f(1) = 2\) and \(f(n+1) = \frac{3f(n)+1}{3}\) for \(n = 1, 2, 3, \ldots\), what is the value of \(f(100)\)?

14. For the family of lines with equations of the form \(px + qy = r\), and which all pass through the point \((-1, 2)\), prove that \(p, q,\) and \(r\) are consecutive terms of an arithmetic sequence.

15. An arithmetic sequence \(S\) has terms \(t_1, t_2, t_3, \ldots\), where \(t_1 = a\) and the common difference is \(d\). The terms \(t_5, t_8,\) and \(t_{16}\) form a three-term geometric sequence with common ratio \(r\). Prove that \(S\) contains an infinite number of three-term geometric sequences, all having the same common ratio \(r\).

16. In the sequence 5, 3, -2, -5, . . . , each term after the first two is constructed by taking the preceding term and subtracting the term before it. What is the sum of the first 32 terms in the sequence?
17. Consider the sequence \( t_1 = 1, t_2 = -1 \) and \( t_n = \left( \frac{n-3}{n-1} \right) t_{n-2} \) where \( n \geq 3 \). What is the value of \( t_{1998} \)?

18. The \( n \)th term of an arithmetic sequence is given by \( t_n = 555 - 7n \). If \( S_n = t_1 + t_2 + \ldots + t_n \), determine the smallest value of \( n \) for which \( S_n < 0 \).