



**Centre for Education  
in Mathematics and Computing**

***Euclid eWorkshop # 6***  
*Solutions*

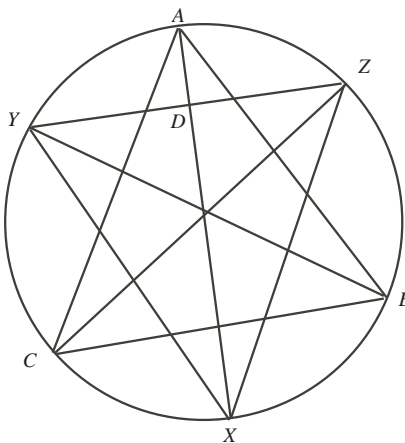
**SOLUTIONS**

1.

$$\begin{aligned}\angle APB &= \angle ACB + \angle CBD \quad (\text{exterior angle of } \triangle BPC) \\ &= \frac{1}{2}\angle AOB + \frac{1}{2}\angle COD.\end{aligned}$$

2. Let  $X$  be the point of intersection of  $PR$  and  $QS$ . From question 1,

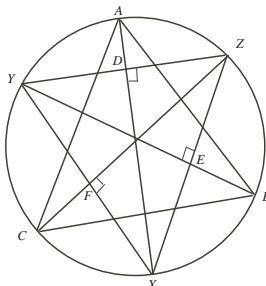
$$\begin{aligned}\angle PXQ &= \frac{1}{2}(\angle POQ + \angle ROS) \\ &= \frac{1}{2}(\angle POB + \angle BOQ + \angle ROD + \angle DOS) \\ &= \frac{1}{4}(\angle AOB + \angle BOC + \angle COD + \angle DOA) \\ &= 90^\circ.\end{aligned}$$

3. The two triangles can be used to find two opposite interior angles in the cyclic quadrilateral, which are  $180^\circ - 5x$  and  $180^\circ - 4x$ . Since these add to  $180^\circ$ , we have  $x = 20^\circ$ .4. We prove first that the bisectors of the angles of  $\triangle ABC$  are altitudes of  $\triangle XYZ$ . Let  $D$  be the point of intersection of  $AX$  and  $YZ$ .

$$\begin{aligned}\text{We know } \angle ACZ + \angle YBC + \angle CAX &= \frac{180^\circ}{2} \quad \text{since these angles are half the angles of } \triangle ABC \\ \angle DXZ + \angle YZC + \angle CZX &= 90^\circ \quad \text{subtended by chords } AZ, YC, CX \text{ respectively} \\ \angle DXZ + \angle DZX &= 90^\circ.\end{aligned}$$

Thus in  $\triangle DZX$ ,  $\angle ZDX = 90^\circ$ . So  $XA$  is an altitude of  $\triangle XYZ$ . A similar argument shows  $BY$  and  $CZ$  are also altitudes.

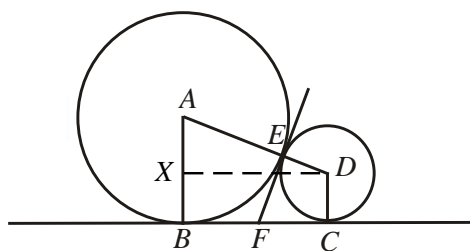
We now prove that the altitudes of  $\triangle XYZ$  bisect the angles of  $\triangle ABC$ .  
 We have with  $AX \perp YZ$  at  $D$ ,  $BY \perp XZ$  at  $E$ ,  $CZ \perp XY$  at  $F$ .



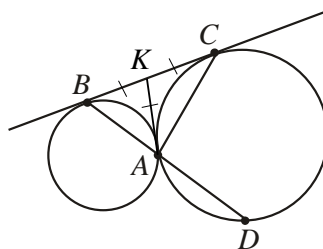
$$\begin{aligned}
 \text{Now } \angle BAX &= \angle BYX \quad (\text{common chord } BX) \\
 &= 90^\circ - \angle ZXY \quad (\triangle EXY) \\
 &= 90^\circ - \angle ZXF \quad (\text{relabel}) \\
 &= \angle XZC \quad (\triangle XZF) \\
 &= \angle XAC \quad (\text{common chord } XC)
 \end{aligned}$$

We have shown that chord  $AX$  bisects  $\angle BAC$ . A similar argument can be used to show that chords  $BY$  and  $CZ$  bisect  $\angle ABC$  and  $\angle ACB$  respectively.

5. Draw the perpendicular  $DX$  from  $D$  to  $AB$ . The triangle  $ADX$  has sides  $AD = 4 + 9 = 13$  and  $XA = 9 - 4 = 5$ . Then  $DX = 12$  using the theorem of Pythagoras. But  $DCBX$  is a rectangle so  $BC = 12$ . But  $FB = FE = FC$  since the 3 segments are tangents from point  $F$ . Thus  $FE = 6$ .



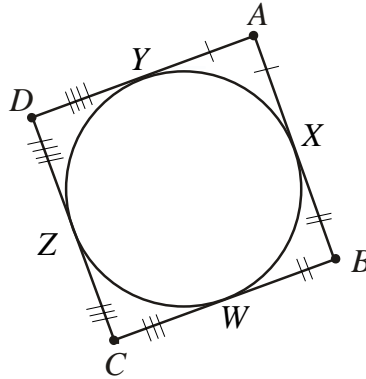
6. (a) Draw the tangent at  $A$  and let this tangent intersect  $BC$  at  $K$ .



Then as in question 5,  $KB = KA = KC$ . Thus a circle drawn on  $BC$  as diameter passes through  $A$  and  $\angle BAC = 90^\circ$ .

- (b) By (a), thus  $\angle CAD = 90^\circ$  and  $CD$  is a diameter.

7. Label the points of tangency  $X$ ,  $Y$ ,  $Z$ , and  $W$  as shown in the diagram.



Then  $AX = AY$ ,  $BX = BW$ ,  $CW = CZ$  and  $DZ = DY$  since all pairs of tangents from an external point are equal. So

$$\begin{aligned} AD + BC &= AY + YD + BW + WC \\ &= AX + DZ + BX + CZ \\ &= AB + CD. \end{aligned}$$

8. Draw the line  $KAL$  to be tangent at  $A$ . Label the points  $X$  and  $Y$  where  $AB$  and  $AC$  intersect the inner circle. We use the Tangent Chord Theorem many times to obtain:

$$\begin{aligned} \angle KAB &= \angle KAX = \angle AYX \quad (\text{TCT in small circle}) \\ &= \angle ADX \quad (\text{common chord } AX) \end{aligned}$$

$$\begin{aligned} \angle LAY &= \angle AXY \quad (\text{TCT in small circle}) \\ &= \angle ABC \quad (\text{TCT in big circle}) \\ &= \angle ABD \end{aligned}$$

$$\angle DAX = \angle XDB \quad (\text{TCT small circle})$$

$$\angle DAY = \angle DXY \quad (\text{common chord } DY)$$

In  $\triangle BAD$  we have

$$\angle BAD + \angle ABD + \angle ADX + \angle XDB = 180^\circ.$$

Replacing equal angles from above gives

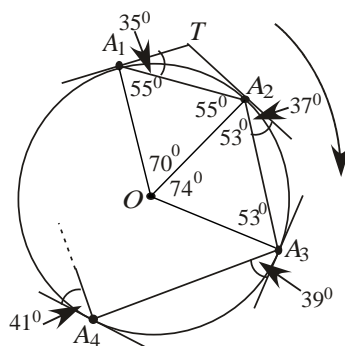
$$\angle BAD + \angle LAY + \angle KAB + \angle DAX = 180^\circ.$$

But since  $KAL$  is a straight line we have

$$\angle BAD + \angle LAY + \angle KAB + \angle YAD = 180^\circ.$$

Thus  $\angle DAX = \angle YAD$  and  $AD$  bisects  $\angle BAC$ .

9. Extend the tangents at  $A_1$  and  $A_2$  to meet at point  $T$  and join  $A_1$  and  $A_2$  to the centre  $O$ . Since tangents from an external point are equal,  $\triangle TA_1A_2$  is isosceles and  $\angle A_1A_2T = \angle A_2A_1T = 35^\circ$ . We now consider  $\triangle OA_1A_2$ . Since the radii are perpendicular to the tangents, we have  $\angle OA_1A_2 = \angle OA_2A_1 = 55^\circ$  and thus  $\angle A_1OA_2 = 70^\circ$ .



Using the same procedure, we get  $\angle A_2OA_3 = 74^\circ$  and in general,

$$\angle A_iOA_{i+1} = [70 + 4(i - 1)]^\circ.$$

(For convenience, we will omit the  $^\circ$  sign; all angles are understood to be in degrees.)

The particle will return to  $A_1$  for the first time when  $\angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + \dots + \angle A_nOA_{n+1}$  is an integer multiple of  $360^\circ$  so that  $A_{n+1} = A_1$ . But

$$\begin{aligned} & \angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + \dots + \angle A_nOA_{n+1} \\ &= 70 + 74 + 78 + \dots + [70 + 4(n - 1)] \\ &= 70n + 4 \frac{(n - 1)(n)}{2} \\ &= 70n + 2n^2 - 2n \\ &= 2n^2 + 68n \end{aligned}$$

So we want a small integer  $k$  such that

$$\begin{aligned} \angle A_1OA_2 + \angle A_2OA_3 + \angle A_3OA_4 + \dots + \angle A_nOA_{n+1} &= 360k \\ 2n^2 + 68n &= 360k \\ n^2 + 34n &= 180k \\ n^2 + 34n + 289 &= 180k + 289 \quad (\text{completing the square}) \\ (n + 17)^2 &= 180k + 289 \end{aligned}$$

Since the left side is a perfect square, the right side must be as well. Trying  $k = 1, 2, 3, \dots$ , we find that  $k = 6$  is the smallest whole value for which the right side is a perfect square. When  $k = 6$ , we have  $n = 20$ .

We can verify that  $70 \cdot 20 + 4 \frac{19 \cdot 20}{2} = 2160$ , which is an integer multiple of 360.