Part A

1. The value of \( \frac{5(6) - 3(4)}{6 + 3} \) is

(A) 1    (B) 2    (C) 6    (D) 12    (E) 31

**Solution**

By evaluating the numerator and denominator we have,

\[
\frac{5(6) - 3(4)}{6 + 3} = \frac{30 - 12}{9} = \frac{18}{9} = 2.
\]

ANSWER: (B)

2. When \( \frac{1}{4} \) of 15 is multiplied by \( \frac{1}{3} \) of 10, the answer is

(A) 5    (B) \( \frac{25}{2} \)    (C) \( \frac{85}{12} \)    (D) \( \frac{99}{8} \)    (E) \( \frac{25}{7} \)

**Solution**

Evaluating gives,

\[
\left[ \frac{1}{4}(15) \right] \left[ \frac{1}{3}(10) \right] = \frac{5}{2} \cdot \frac{5}{3} = \frac{25}{6} \cdot \frac{5}{3} = \frac{25}{2}
\]

ANSWER: (B)

3. If \( x = \frac{1}{4} \), which of the following has the largest value?

(A) \( x \)    (B) \( x^2 \)    (C) \( \frac{1}{2}x \)    (D) \( \frac{1}{x} \)    (E) \( \sqrt{x} \)

**Solution**

If we calculate the value of the given expressions, we get

\[
\begin{align*}
(A) & \quad \frac{1}{4} = \frac{1}{16} \\
(B) & \quad \left( \frac{1}{4} \right)^2 = \frac{1}{8} \\
(C) & \quad \frac{1}{2} \left( \frac{1}{4} \right) = \frac{1}{8} \\
(D) & \quad \frac{1}{x} = 4 \\
(E) & \quad \sqrt{x} = \frac{1}{2}
\end{align*}
\]

ANSWER: (D)

4. In a school, 30 boys and 20 girls entered the Cayley competition. Certificates were awarded to 10% of the boys and 20% of the girls. Of the students who participated, the percentage that received
certificates was

(A) 14  (B) 15  (C) 16  (D) 30  (E) 50

Solution
If 30 boys entered the Cayley competition and 10% of them received certificates, this implies that (0.1)(30) or 3 boys received certificates. Of the 20 girls who entered the competition (0.2)(20) or 4 girls received certificates. This implies that 7 students in total out of 50 received certificates.

Thus 14% of the students in total received certificates.  

ANSWER:  (A)

5. In the diagram, \( KL \) is parallel to \( MN \), \( AB = BC \), and \( \angle KAC = 50^\circ \). The value of \( x \) is

(A) 40  (B) 65  (C) 25  (D) 100  (E) 80

Solution
Since \( KL \) is parallel to \( MN \), \( \angle ACB \) and \( \angle KAC \) are alternate angles. Thus, \( \angle ACB = 50^\circ \). We are given that \( \triangle BCA \) is isosceles, so \( \angle BCA = \angle BAC = 50^\circ \). We know that since angles \( KAC \), \( BAC \) and \( LAB \) have a sum of 180°,

\[
50^\circ + 50^\circ + x^\circ = 180^\circ
\]

\[
x = 80.
\]

ANSWER:  (E)

6. Dean scored a total of 252 points in 28 basketball games. Ruth played 10 fewer games than Dean. Her scoring average was 0.5 points per game higher than Dean’s scoring average. How many points, in total, did Ruth score?

(A) 153  (B) 171  (C) 180  (D) 266  (E) 144

Solution
If Dean scored 252 points in 28 games this implies that he averages \( \frac{252}{28} \) or 9 points per game.

Ruth must then have averaged 9.5 points in each of the 18 games she played. In total she scored

\( 9.5 \times 18 \) or 171 points.

ANSWER:  (B)
7. In the diagram, square $ABCD$ has side length 2, with $M$ the midpoint of $BC$ and $N$ the midpoint of $CD$. The area of the shaded region $BMND$ is

(A) $1$  
(B) $2\sqrt{2}$  
(C) $\frac{4}{3}$  
(D) $\frac{3}{2}$  
(E) $4 - \frac{3}{2}\sqrt{2}$

Solution

The area of $\triangle MNC$ is $\frac{1}{2}(1)(1) = \frac{1}{2}$. Since $\triangle BDC$ is half the square, it will have an area of 2.

Since the shaded region has an area equal to that of $\triangle BDC$ minus the area of $\triangle MNC$, its area will be $2 - \frac{1}{2} = \frac{3}{2}$.

ANSWER: (D)

8. The line $L$ crosses the $x$-axis at $(-8, 0)$. The area of the shaded region is 16. What is the slope of the line $L$?

(A) $\frac{1}{2}$  
(B) $4$  
(C) $-\frac{1}{2}$  
(D) $2$  
(E) $-2$

Solution

If the area of the shaded region is 16 and its base has a length of 8, its height must then be 4.

Thus we have the changes noted in the diagram.

Thus the slope is $\frac{4 - 0}{0 - (-8)} = \frac{1}{2}$ or $\frac{1}{2}$ because the line slopes down from right to left and the line has a rise of 4 and a run of 8.

ANSWER: (A)

9. If $\left[(10^3)(10^x)\right]^2 = 10^{18}$, the value of $x$ is

(A) $\sqrt{2}$  
(B) 12  
(C) 6  
(D) 1  
(E) 3
Solution
If we simplify using laws of exponents, we will have,
\[ (10^{3+x})^2 = 10^{18} \]
\[ 10^{6+2x} = 10^{18} \]
In order that the left and right side be equal, it is necessary that exponents be equal.
Thus, \( 6 + 2x = 18 \)
\[ 2x = 12 \]
\[ x = 6. \]

ANSWER: (C)

10. The sum of five consecutive integers is 75. The sum of the largest and smallest of these five integers is

(A) 15  (B) 25  (C) 26  (D) 30  (E) 32

Solution
There are a variety of ways of approaching this problem. The easiest is to represent the integers as \( x-2, x-1, x, x+1, \) and \( x+2. \)
Thus, \( (x-2) + (x-1) + x + (x+1) + (x+2) = 75 \)
\[ 5x = 75 \]
\[ x = 15. \]
The five consecutive integers are 13, 14, 15, 16, and 17.
The required sum is \( 13 + 17 = 30. \)

ANSWER: (D)

Part B

11. When a positive integer \( N \) is divided by 60, the remainder is 49. When \( N \) is divided by 15, the remainder is

(A) 0  (B) 3  (C) 4  (D) 5  (E) 8

Solution
This problem can be done in a number of ways. The easiest way is to consider that if \( N \) is divided by 60 to achieve a remainder of 49, it must be a number of the form, \( 60k + 49, \) \( k = 0, 1, 2, \ldots. \)
This implies that the smallest number to meet the requirements is 49 itself. If we divide 49 by 15 we get a remainder of 4. Or, if \( k = 1 \) in our formula then the next number to satisfy the requirements is 109 which when divided by 15 gives 4 as the remainder.

ANSWER: (C)

12. The 6 members of an executive committee want to call a meeting. Each of them phones 6 different
people, who in turn each calls 6 other people. If no one is called more than once, how many people will know about the meeting?

(A) 18  (B) 36  (C) 216  (D) 252  (E) 258

Solution
If 6 people each call 6 other people in the first round of calls, there will be 36 people making 6 calls each for an additional 216 calls. Altogether, there will be the original 6, followed by 36 who in turn phone another 216.
In total, there are $6 + 36 + 216 = 258$.

Answer: (E)

13. The sequences 3, 20, 37, 54, 71, … and 16, 27, 38, 49, 60, 71, … each have 71 as a common term. The next term that these sequences have in common is

(A) 115  (B) 187  (C) 258  (D) 445  (E) 1006

Solution
The first sequence increases by a constant value of 17 and the second by a constant value of 11. After 71, the next common term will be 71 plus the Lowest Common Multiple of 11 and 17. Since the L.C.M. of 11 and 17 is 187 the next term will be $71 + 187 = 258$.

Answer: (C)

14. In the rectangle shown, the value of $a - b$ is

(A) $-3$  (B) $-1$  (C) 0  (D) 3  (E) 1

Solution
To go from the point $(5, 5)$ to the point $(9, 2)$ we must move over 4 and down 3. Since we are dealing with a rectangle, the same must be true for $(a, 13)$ and $(15, b)$. Thus, $a + 4 = 15$ and $13 - 3 = b$. From this, $a = 11$ and $b = 10$. So $a - b = 11 - 10 = 1$.

Answer: (E)

15. A small island has $\frac{2}{5}$ of its surface covered by forest and $\frac{1}{4}$ of the remainder of its surface by sand dunes. The island also has 90 hectares covered by farm land. If the island is made up of only forest, sand dunes and farm land, what is the total area of the island, to the nearest hectare?

(A) 163  (B) 120  (C) 200  (D) 138  (E) 257
Solution
If $\frac{2}{5}$ of an island is covered by forest then $\frac{3}{5}$ of the island is made up of sand dunes and farm land.

Since $\frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$ is made up of sand dunes this implies that $\frac{2}{5} + \frac{3}{20} = \frac{11}{20}$ of the island is made up of forest and sand dunes. Thus $\frac{9}{20}$ of the island, or 90 hectares, is made up of farm land.

Thus, the island must be $\frac{20}{9} = 20\overline{9}$ or 200 hectares in total. ANSWER: (C)

16. How many integer values of $x$ satisfy $\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$?

(A) 0        (B) 1        (C) 2        (D) 3        (E) 4

Solution
If we multiply all three fractions by $3(5)(7)$ we have,

\[
(3)(5)(7) \left( \frac{x-1}{3} \right) < (3)(5)(7) \left( \frac{5}{7} \right) < (3)(5)(7) \left( \frac{x+4}{5} \right)
\]

\[
35(x-1) < 75 < 35(x+4)
\]

In order to satisfy this inequality then,

\[
35(x-1) < 75 \quad \text{and} \quad 21(x+4) > 75
\]

\[
35x - 35 < 75 \quad \text{and} \quad 21x + 84 > 75
\]

\[
x < \frac{110}{35} \quad \text{and} \quad x > -\frac{9}{21}
\]

\[
x < 3 \frac{1}{7} \quad \text{and} \quad x > -\frac{9}{21}
\]

The only integers to satisfy both conditions are then in the set $\{0, 1, 2, 3\}$. ANSWER: (E)

17. $ABCDEFGH$ is a cube having a side length of 2. $P$ is the midpoint of $EF$, as shown. The area of $\triangle APB$ is

(A) $\sqrt{8}$        (B) 3        (C) 6

(D) $\sqrt{2}$        (E) $\sqrt{32}$

[Diagram of cube with midpoint and triangle]
Solution
By symmetry, the lengths of $AP$ and $BP$ will be equal, and
$$AP = \sqrt{AD^2 + DE^2 + EP^2} = \sqrt{2^2 + 2^2 + 1^2} = 3.$$ 
If $M$ is the midpoint of $AB$, then $PM$ is perpendicular to $AB$. By Pythagoras, $MP = \sqrt{3^2 - 1^2} = \sqrt{8}$. 
So the area of $\triangle APB$ is
$$\text{Area} = \frac{1}{2}(2\sqrt{8}) = \sqrt{8}.$$ 

18. How many five-digit positive integers, divisible by 9, can be written using only the digits 3 and 6?
(A) 5  (B) 2  (C) 12  (D) 10  (E) 8

Solution
If a five-digit number is composed of only 3’s and 6’s, there are just two cases to consider because these are the only ways that the digital sum can be a multiple of 9.

Case 1  one 6, four 3’s
In this case, there are five numbers: 63333, 36333, 33633, 33363 and 33336.

Case 2  one 3, four 6’s
In this case there are five numbers: 36666, 63666, 66366, 66636 and 66663.

In total there are 10 possible numbers.  

19. Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is
(A) 34  (B) 35  (C) 36  (D) 37  (E) 38

Solution
Let the three numbers be $a$, $b$ and $c$.
We construct the first equation to be,
$$a + \frac{b + c}{2} = 65.$$ 
Or, $2a + b + c = 130$.
Similarly we construct the two other equations to be,
$$a + 2b + c = 138$$
and $a + b + 2c = 152$.
If we add the three equations we obtain,
\[ 4a + 4b + 4c = 420. \]
The average is \[ \frac{4(a + b + c)}{12} = \frac{420}{12}. \]
Or, \[ \frac{a + b + c}{3} = 35. \]  
\text{Answer: (B)}

20. Square \(ABCD\) with side length 2 is inscribed in a circle, as shown. Using each side of the square as a diameter, semi-circular arcs are drawn. The area of the shaded region outside the circle and inside the semi-circles is

(A) \(\pi\) \hspace{1cm} (B) 4 \hspace{1cm} (C) \(2\pi - 2\) \hspace{1cm} (D) \(\pi + 1\) \hspace{1cm} (E) \(2\pi - 4\)

\textbf{Solution}

The side length of the square is 2 and thus the diameter of the circle has length \(2\sqrt{2}\) which is also the length of the diagonal \(AC\) (or \(BD\)). The area of the circle is thus \(\pi \left(\sqrt{2}\right)^2 = 2\pi\). Since the side length of the square is 2, it will have an area of 4. From this, we calculate the area of the circle outside the square to be \(2\pi - 4\). To calculate the shaded area, we first calculate the area of each semi-circle. Each of the semi-circles has a radius of 1 meaning that each semi-circle will have an area of \(\frac{1}{2} \pi (1)^2 = \frac{1}{2} \pi\). In total, the four semi-circles have an area of \(2\pi\). Thus the shaded area has an area of \(2\pi - (2\pi - 4) = 4\).

\text{Answer: (B)}

\textbf{Part C}

21. Point \(P\) is on the line \(y = 5x + 3\). The coordinates of point \(Q\) are \((3, -2)\). If \(M\) is the midpoint of \(PQ\), then \(M\) must lie on the line

(A) \(y = \frac{5}{2} x - \frac{7}{2}\) \hspace{1cm} (B) \(y = 5x + 1\) \hspace{1cm} (C) \(y = -\frac{1}{5} x - \frac{7}{5}\) \hspace{1cm} (D) \(y = \frac{5}{2} x + \frac{1}{2}\) \hspace{1cm} (E) \(y = 5x - 7\)
We start by drawing a diagram and labelling the intercepts.

**Solution 1**
Since the point $P$ is on the line $y = 5x + 3$, select $P(0, 3)$ as a point on this line.
The midpoint of $PQ$ is $M\left(\frac{3+0}{2}, \frac{-2+3}{2}\right) = M\left(\frac{3}{2}, \frac{1}{2}\right)$.
The required line must contain $M$ and be midway between the given point and $y = 5x + 3$. The only possible line meeting this requirement is the line containing $M\left(\frac{3}{2}, \frac{1}{2}\right)$ and which has a slope of 5. The required line will have as its equation
\[ y - \frac{1}{2} = 5\left(x - \frac{3}{2}\right) \]
or, \[ y = 5x - 7. \]

**Solution 2**
Let a general point on the line $y = 5x + 3$ be represented by $(a, 5a + 3)$. Also, let a point on the required line be $M(x, y)$. Since $M(x, y)$ is the midpoint of $PQ$ then
\[ (1) \quad x = \frac{a + 3}{2} \quad \text{and} \quad (2) \quad y = \frac{(5a + 3) + (-2)}{2} \]
\[ y = \frac{5a + 1}{2} \]
Solving (1) for $a$, we have $a = 2x - 3$ and solving (2) for $a$, we have $\frac{2y - 1}{5} = a$.
Equating gives, $2x - 3 = \frac{2y - 1}{5}$
\[ 10x - 15 = 2y - 1 \]
or, \[ y = 5x - 7. \] 

22. What is the shortest distance between two circles, the first having centre $A(5, 3)$ and radius 12, and the other with centre $B(2, -1)$ and radius 6?

(A) 1 \quad (B) 2 \quad (C) 3 \quad (D) 4 \quad (E) 5
Solution
We start by drawing the two circles where the larger circle has centre \( A(5, 3) \) and the smaller circle has centre \( B(2, -1) \). A line is drawn from \( A \), through \( B \) to meet the circumference of the smaller circle at \( C \) and the circumference of the larger circle at \( D \). The length \( CD \) is the desired length. The length from \( A \) to \( D \) is given to be 12 and the length from \( B \) to \( C \) is 6. We calculate the length of \( AB \) to be, \( \sqrt{[3 - (-1)]^2 + (5 - 2)^2} = \sqrt{16 + 9} = 5 \). To find \( CD \), we calculate as follows,
\[
CD = AD - (AB + BC) \\
= 12 - (5 + 6) \\
= 1.
\]

ANSWER: (A)

23. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

\[
\begin{align*}
\text{Figure A} & \quad \text{Figure B} \\
\text{height of liquid} & \quad 20 \text{ cm}
\end{align*}
\]

\[
\begin{align*}
\text{Figure C} & \\
& \quad 28 \text{ cm}
\end{align*}
\]

(A) 29 \hspace{1cm} (B) 30 \hspace{1cm} (C) 31 \hspace{1cm} (D) 32 \hspace{1cm} (E) 48

Solution
We’ll start by representing the height of the large cylinder as \( h_1 \) and the height of the small cylinder as \( h_2 \). For simplicity, we’ll let \( x = h_1 + h_2 \).

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to \( x - 20 \) and a volume equal to, \( \pi(1)^2(x - 20) \).

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a
height equal to \( x - 28 \) and a volume equal to, \( \pi(3)^2(x - 28) \).

Since these two unoccupied spaces must be equal, we then have,

\[
\pi(1)^2(x - 20) = \pi(3)^2(x - 28)
\]

\[
x - 20 = 9x - 252
\]

\[
8x = 272
\]

\[
x = 29.
\]

Therefore, the total height is 29.

**Answer:** (A)

24. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

(A) 28  (B) 32  (C) 36  (D) 40  (E) 44

**Solution**

Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome then the following must be true,

\[
\begin{array}{c c c c c}
\text{a} & \text{b} & \text{b} & \text{a} \\
\text{c} & \text{d} & \text{d} & \text{c} \\
\hline
1 & \text{e} & \text{f} & \text{e} & 1
\end{array}
\]

(i.e. the first digit of the 5-digit palindrome is 1.)

From this, we can see that \( a + c = 11 \) since \( a + c \) has a units digit of 1 and \( 10 < a + c < 20 \).

We first note that there are four possibilities for \( a \) and \( c \). We list the possibilities:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

Note that there are only four possibilities here.

(If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with \( a \) and \( c \) reversed.)

Let us consider one case, say \( a = 2 \) and \( c = 9 \).

\[
\begin{array}{c c c c c}
2 & \text{b} & \text{b} & 2 \\
9 & \text{d} & \text{d} & 9 \\
\hline
1 & \text{e} & \text{f} & \text{e} & 1
\end{array}
\]

From this, we can only get palindromes in two ways. To see this we note that \( e \) is either 1 or 2 depending on whether we get a carry from the previous column (we see this looking at the thousands digit of \( 1 \text{e} \text{f} \text{e} \text{1} \)). If \( e = 1 \), then \( b + d \) has no carry and so looking at the tens digit \( e = 1 \), we see that \( b + d = 0 \) to get this digit.
If \( e = 2 \), we do get a carry from \( b + d \), so looking again at the tens digit \( e = 2 \), we see that \( b + d = 11 \).

**Possibility 1 \( b = d = 0 \)**

Since there are only four possibilities for \( a \) and \( c \) and just one way of selecting \( b \) and \( d \) so that \( b + d = 0 \) for each possibility, there are just four possibilities.

**Possibility 2 \( b + d = 11 \)**

For each of the four possible ways of choosing \( a \) and \( c \), there are eight ways of choosing \( b \) and \( d \) so that \( b + d = 11 \) thus giving 32 possibilities.

This gives a total of \( 4 + 32 = 36 \) possibilities.

**ANSWER: (C)**

25. The circle with centre \( A \) has radius 3 and is tangent to both the positive \( x \)-axis and positive \( y \)-axis, as shown. Also, the circle with centre \( B \) has radius 1 and is tangent to both the positive \( x \)-axis and the circle with centre \( A \). The line \( L \) is tangent to both circles. The \( y \)-intercept of line \( L \) is

(A) \( 3 + 6\sqrt{3} \)  
(B) \( 10 + 3\sqrt{2} \)  
(C) \( 8\sqrt{3} \)  
(D) \( 10 + 2\sqrt{3} \)  
(E) \( 9 + 3\sqrt{3} \)

**Solution**

We start by drawing a line from point \( C \) that will pass through \( A \) and \( B \). From \( A \) and \( B \), we drop perpendiculars to the points of tangency on the \( x \)-axis and label these points as \( E \) and \( F \) as shown. We also drop a perpendicular from \( A \) to the \( y \)-axis which makes \( AH = AE = 3 \).
Extracting $\triangle CAE$ from the diagram and labelling with the given information we would have the following noted in the diagram.

If we represent the distance from $C$ to $B$ as $x$ and recognize that $\triangle CBF$ is similar to $\triangle CAE$,

$$\frac{x}{1} = \frac{x + 4}{3}$$

$$x = 2.$$\

In $\triangle CBF$, $FC^2 = 2^2 - 1^2 = 3$

$$FC = \sqrt{3}, \ (FC > 0).$$

This implies that $\angle BCF = 30^\circ$ and $\angle OCD = 60^\circ$. Therefore $EF = 2\sqrt{3}$, from similar triangles again.

This now gives us the diagram shown.

Thus, $d = \sqrt{3}(3 + 3\sqrt{3})$

$$= 3\sqrt{3} + 9.$$