Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2002 Solutions
Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) *Solution 1 (Midpoint Formula)*

Since M is the midpoint of the line segment joining R and S, then looking at the x-coordinate of M,

$$7 = \frac{1+a}{2}$$
$$14 = 1 + a$$
$$a = 13$$

Solution 2 (Slopes)

Since the slope of RM is equal to the slope of MS, then

$$\frac{3}{6} = \frac{3}{a-7}$$
$$a - 7 = 6$$
$$a = 13$$

Solution 3 (Distances)

Since $RM = MS$ or $RM^2 = MS^2$, then

$$36 = 3^2 + (a-7)^2$$
$$0 = a^2 - 14a + 13$$
$$0 = (a-13)(a-1)$$

Therefore, $a = 13$ or $a = 1$, but we reject $a = 1$, since $(1,10)$ does not lie on the line.

Thus, $a = 13$.

Answer: $a = 13$

(b) The base of ΔPQR has length 8, and the height has length $k-2$ (since $k > 0$).

Since the area of ΔPQR is 24, then

$$\tfrac{1}{2}(8)(k-2) = 24$$
$$4k - 8 = 24$$
$$4k = 32$$
$$k = 8$$

Answer: $k = 8$

(c) We first determine the point of intersection of lines $y = 2x + 3$ and $y = 8x + 15$, and then substitute this point into the line $y = 5x + b$, since it lies on all three lines.

So we set the first two equations equal to each other:

$$
\begin{aligned}
2x + 3 &= 8x + 15 \\
-12 &= 6x \\
x &= -2
\end{aligned}
$$

Substituting $x = -2$ into the first equation, we obtain $y = 2(-2) + 3 = -1$, so the point of intersection is $(-2, -1)$, which must lie on the third line.

Thus,

$$
\begin{aligned}
-1 &= 5(-2) + b \\
b &= 9
\end{aligned}
$$

Therefore, the value of b is 9.

2. (a) *Solution 1*

Since $x = 4$ is a root, then $4^2 - 3(4) + c = 0$ or $c = -4$.

Therefore, the quadratic equation is $x^2 - 3x - 4 = 0$, which we can factor as $(x-4)(x+1) = 0$. (This factorization is made easier since we already know one of the roots.) Therefore, the second root is $x = -1$.

Solution 2

The sum of the roots of $x^2 - 3x + c = 0$ is $-\left(\frac{-3}{1}\right) = 3$, so since one root is 4, the second root must be $x = -1$.

Answer: $x = -1$

(b) *Solution 1*

Since the two expressions are the same, then they must have the same value when we substitute any value for *x*. In particular, substitute $x = 2$, and so we get

$$
\begin{aligned}
\frac{2\left(2^2\right)+1}{2^2-3} &= 2 + \frac{A}{2^2-3} \\
9 &= 2 + A \\
A &= 7
\end{aligned}
$$

Solution 2

We compare the two expressions

$$
\begin{aligned}
\frac{2x^2+1}{x^2-3} &= 2 + \frac{A}{x^2-3} \\
&= \frac{2\left(x^2-3\right)}{x^2-3} + \frac{A}{x^2-3} \\
&= \frac{2x^2-6+A}{x^2-3}
\end{aligned}
$$

Since the expressions are the same, the numerators must be the same, and so $-6+A=1$ or $A=7$.

Solution 3

$$\frac{2x^2+1}{x^2-3}=\frac{2x^2-6+7}{x^2-3}$$
$$=\frac{2\left(x^2-3\right)+7}{x^2-3}$$
$$=2+\frac{7}{x^2-3}$$

Therefore, $A=7$.

Answer: $A=7$

(c) *Solution 1*

The original parabola can be written as
$$y=(x-3)(x-1)$$
which means its roots are $x=3$ and $x=1$.
When this parabola is shifted 5 units to the right, the parabola obtained must thus have roots $x=3+5=8$ and $x=1+5=6$.
Therefore, the new parabola is
$$y=(x-8)(x-6)$$
$$=x^2-14x+48$$
and so $d=48$.

Solution 2

The original parabola $y=x^2-4x+3$ can be written as $y=(x-2)^2-1$, and so its vertex has coordinates $(2,-1)$. To get the vertex of the new parabola, we shift the vertex of the original parabola 5 units to the right to the point $(7,-1)$. Substituting this point into the new parabola, we obtain
$$-1=7^2-14(7)+d$$
$$-1=49-98+d$$
$$d=48$$
[An easier version of this solution is to recognize that if the original parabola passes through $(1,0)$, then $(6,0)$ must be on the translated parabola. Thus, $0=36-14(6)+d$ or $d=48$, as above.]

Solution 3

To carry out a translation of 5 units to the right, we can define new coordinates X and Y, with $(x, y) = (X - 5, Y)$. So in these new coordinates the parabola will have equation

$$\begin{aligned} Y &= (X-5)^2 - 4(X-5) + 3 \\ &= X^2 - 10X + 25 - 4X + 20 + 3 \\ &= X^2 - 14X + 48 \end{aligned}$$

Comparing this with the given equation, we see that $d = 48$.

3. (a) We make a table of the possible selections of balls a, b, c that give $a = b + c$:

a	b	c
2	1	1
3	1	2
3	2	1
4	1	3
4	2	2
4	3	1

Therefore, since there are 6 ways to get the required sum, then the probability that he wins the prize is $\frac{6}{64} = \frac{3}{32}$.

Answer: $\frac{3}{32}$

(b) Since the product of the three integers is 216, then

$$\begin{aligned} a(ar)\left(ar^2\right) &= 216 \\ a^3r^3 &= 216 \\ (ar)^3 &= 6^3 \\ ar &= 6 \end{aligned}$$

Now we are given that a is a positive integer, but r is not necessarily an integer. However, we do know that the sequence is *increasing*, so $r > 1$, and thus $a < 6$. We check the possibilities for a between 1 and 5, and determine whether these possibilities for a yield a value for a that makes the third term (ar^2) an integer (we already know that $ar = 6$, so is an integer):

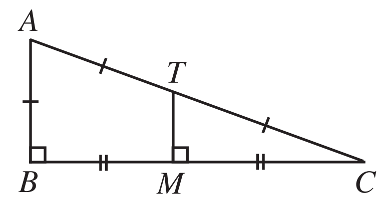
a	r	ar	ar^2
1	6	6	36
2	3	6	18
3	2	6	12

4	$\frac{3}{2}$	6	9
5	$\frac{6}{5}$	6	$\frac{36}{5}$

So the four sequences that satisfy the required conditions are:

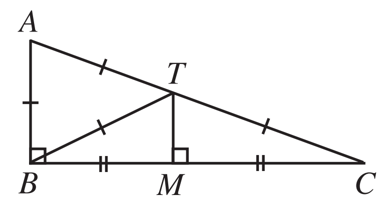
1, 6, 36;
2, 6, 18;
3, 6, 12;
4, 6, 9.

4. (a) *Solution 1*



Since MT is the perpendicular bisector of BC, then $BM = MC$, and TM is perpendicular to BC.
Therefore, ΔCMT is similar to ΔCBA, since they share a common angle and each have a right angle.
But $\frac{CM}{CB} = \frac{1}{2}$ so $\frac{CT}{CA} = \frac{CM}{CB} = \frac{1}{2}$, and thus $CT = AT = AB$, ie. $\frac{AB}{AC} = \frac{1}{2}$ or $\sin(\angle ACB) = \frac{1}{2}$.
Therefore, $\angle ACB = 30^\circ$.

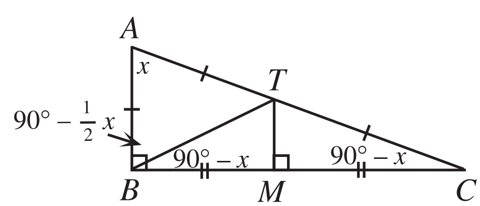
Solution 2



Since $TM \| AB$, and $CM = MB$, then $CT = TA = AB$.
Join T to B.
Since $\angle ABC = 90^\circ$, then AC is the diameter of a circle passing through A, C and B, with T as its centre.

Thus, $TA = AB = BT$ (all radii), and so ΔABT is equilateral. Therefore, $\angle BAC = 60^\circ$, and so $\angle ACB = 30^\circ$.

Solution 3



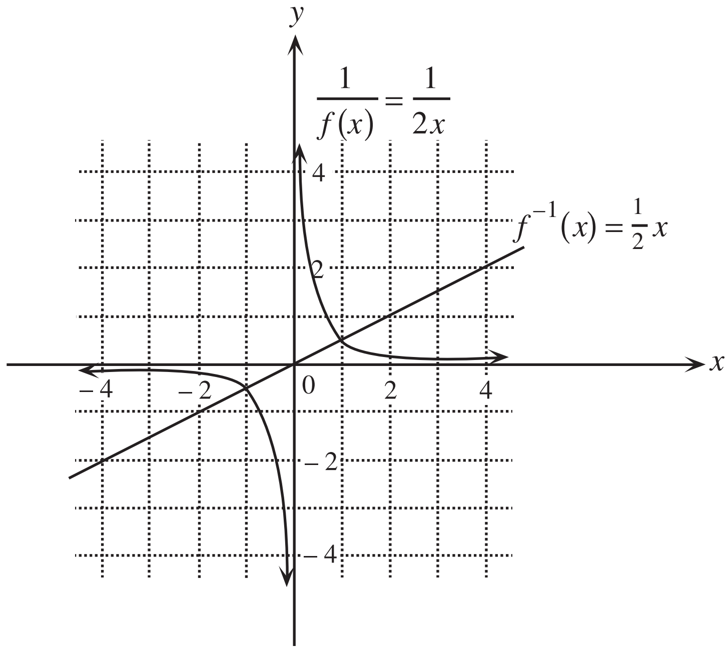
Join T to B, and let $\angle BAC = x$. Thus, $\angle ACB = 90^\circ - x$
As in Solution 1 or Solution 2, ΔATB is isosceles, so $\angle ABT = 90^\circ - \frac{1}{2}x$.
Since ΔTBM is congruent to ΔTCM (common side; right angle; equal side), then $\angle TBC = \angle ACB = 90^\circ - x$
Looking at $\angle ABC$, we see that

$$\left(90^\circ - \tfrac{1}{2}x\right) + (90^\circ - x) = 90^\circ$$
$$90^\circ = \tfrac{3}{2}x$$
$$x = 60^\circ$$

Therefore, $\angle ACB = 30^\circ$.

Answer: $\angle ACB = 30^\circ$

(b) (i)



(ii) *Solution 1*

From the graph in (i), the points where $f^{-1}(x) = \dfrac{1}{f(x)}$ are $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.

Solution 2

We determine the functions $f^{-1}(x)$ and $\dfrac{1}{f(x)}$ explicitly.

To get $f^{-1}(x)$, we start with $y = 2x$, interchange x and y to get $x = 2y$, and solve for y to get $y = \frac{1}{2}x$ or $f^{-1}(x) = \frac{1}{2}x$.

Also, $\dfrac{1}{f(x)} = \dfrac{1}{2x}$.

Setting these functions equal,

$$\frac{1}{2}x = \frac{1}{2x}$$
$$x^2 = 1$$
$$x = \pm 1$$

Substituting into $f^{-1}(x)$, we obtain the points $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.

(iii) Using $f(x) = 2x$, we see that $f\left(\frac{1}{2}\right) = 1$, and so

$$\begin{aligned} f^{-1}\left(\frac{1}{f\left(\frac{1}{2}\right)}\right) &= f^{-1}\left(\frac{1}{1}\right) \\ &= f^{-1}(1) \\ &= \tfrac{1}{2}(1) \\ &= \tfrac{1}{2} \end{aligned}$$

where $f^{-1}(1)$ is determined either from the explicit form of $f^{-1}(x)$ or from the graph.

5. (a) Combining the logarithms,

$$\begin{aligned} \log_5(x+3)+\log_5(x-1) &= 1 \\ \log_5\left((x+3)(x-1)\right) &= 1 \\ \log_5\left(x^2+2x-3\right) &= 1 \\ x^2+2x-3 &= 5 \\ x^2+2x-8 &= 0 \\ (x+4)(x-2) &= 0 \end{aligned}$$

Therefore, $x=-4$ or $x=2$. Substituting the two values for x back into the original equation, we see that $x=2$ works, but that $x=-4$ does not, since we cannot take the logarithm of a negative number.

Answer: $x=2$

(b) (i) From the table we have two pieces of information, so we substitute both of these into the given formula.

$$2.75 = a(3.00)^b$$
$$3.75 = a(6.00)^b$$

We can now proceed in either of two ways to solve for b.

Method 1 to find b

Dividing the second equation by the first, we obtain

$$\frac{3.75}{2.75} = \frac{a(6.00)^b}{a(3.00)^b} = \frac{(6.00)^b}{(3.00)^b} = \left(\frac{6.00}{3.00}\right)^b = 2^b$$

or

$$2^b \approx 1.363636$$

Taking logarithms of both sides,

$$\log\left(2^b\right) \approx \log(1.363636)$$
$$b\log(2) \approx \log(1.363636)$$
$$b \approx \frac{\log(1.363636)}{\log(2)}$$
$$b \approx 0.4475$$

Method 2 to find b

Taking logarithms of both sides of the above equations, we obtain

$$\begin{aligned}\log(2.75) &= \log\left(a(3.00)^b\right)\\ &= \log(a) + \log\left((3.00)^b\right)\\ &= \log(a) + b\log(3.00)\end{aligned}$$

Similarly,

$$\log(3.75) = \log(a) + b\log(6.00)$$

Subtracting the first equation from the second, we obtain

$$\log(3.75) - \log(2.75) = b(\log(6.00) - \log(3.00))$$
$$b = \frac{\log(3.75) - \log(2.75)}{\log(6.00) - \log(3.00)}$$
$$b \approx 0.4475$$

We now continue in the same way for both methods.

Substituting this value for b back into the first equation above,

$$2.75 \approx a(3.00)^{0.4475}$$
$$a \approx \frac{2.75}{(3.00)^{0.4475}}$$
$$a \approx 1.6820$$

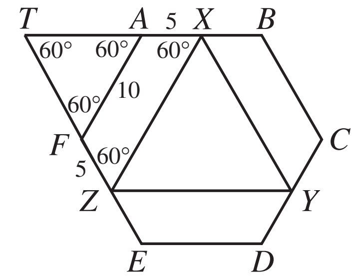
Therefore, to two decimal places, $a = 1.68$ and $b = 0.45$.

(ii) To determine the time to cook a goose of mass 8.00 kg, we substitute $m = 8.00$ into the given formula:

$$\begin{aligned}t &= am^b\\ &\approx 1.68(8.00)^{0.45}\\ &\approx 4.2825\end{aligned}$$

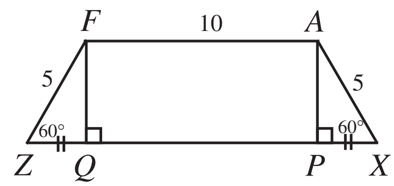
Thus, it will take about 4.28 h until his goose is cooked.

6. (a) *Solution 1*



Extend XA and ZF to meet at point T.
By symmetry, $\angle AXZ = \angle FZX = 60^\circ$ and $\angle TAF = \angle TFA = 60^\circ$, and so ΔTAF and ΔTXZ are both equilateral triangles.
Since $AF = 10$, then $TA = 10$, which means $TX = 10 + 5 = 15$, and so $XZ = TX = 15$.

Solution 2



We look at the quadrilateral $AXZF$.
Since $ABCDEF$ is a regular hexagon, then $\angle FAX = \angle AFZ = 120^\circ$.
Note that $AF = 10$, and also $AX = FZ = 5$ since X and Z are midpoints of their respective sides.
By symmetry, $\angle AXZ = \angle FZX = 60^\circ$, and so $AXZF$ is a trapezoid.
Drop perpendiculars from A and F to P and Q, respectively, on XZ.
By symmetry again, $PX = QZ$. Now, $PX = AX\cos 60^\circ = 5\left(\frac{1}{2}\right) = \frac{5}{2}$.
Since $APQF$ is a rectangle, then $PQ = 10$.
Therefore, $XZ = XP + PQ + QZ = \frac{5}{2} + 10 + \frac{5}{2} = 15$.

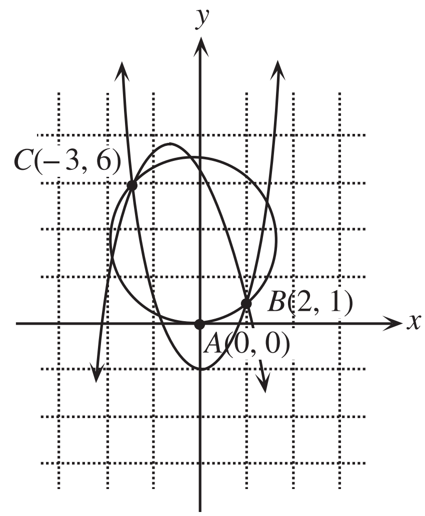
Answer: $XZ = 15$

(b) We first determine the three points through which the circle passes.
The first point is the origin $(0,0)$.
The second and third points are found by determining the points of intersection of the two parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. We do this by setting the y values equal.

$$\begin{aligned} x^2 - 3 &= -x^2 - 2x + 9 \\ 2x^2 + 2x - 12 &= 0 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \end{aligned}$$

so $x = -3$ or $x = 2$.

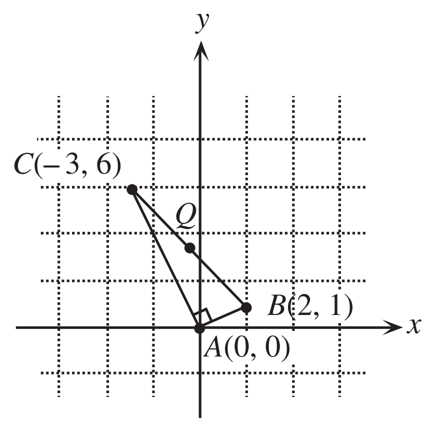
We determine the points of intersection by substituting into the first parabola.
If $x = 2$, $y = 2^2 - 3 = 1$, so the point of intersection is $(2,1)$.
If $x = -3$, $y = (-3)^2 - 3 = 6$, so the point of intersection is $(-3,6)$.
Therefore, the circle passes through the three points $A(0,0)$, $B(2,1)$ and $C(-3,6)$.
Let the centre of the circle be the point $Q(a,b)$.



Finding the centre of the circle can be done in a variety of ways.

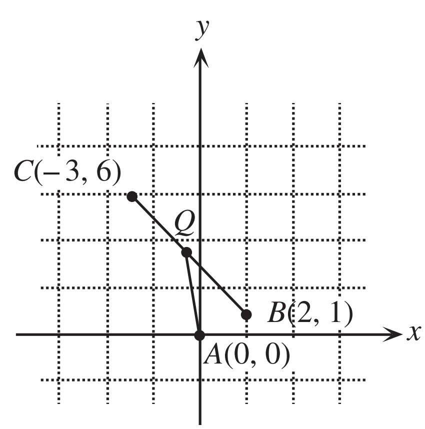
Method 1 $\left(\angle CAB = 90^\circ\right)$

We notice that the line segment joining $A(0,0)$ to $B(2,1)$ has slope $\frac{1}{2}$, and the line segment joining $A(0,0)$ to $C(-3,6)$ has slope -2, and so the two lines are perpendicular (since $\frac{1}{2}(-2) = -1$). Therefore, $\angle BAC = 90^\circ$.
Since BC is a chord of the circle which subtends an angle of 90° at point A on the circle, then BC is a diameter of the circle. Therefore, the centre of the circle is the midpoint of BC, which is the point $\left(-\frac{1}{2}, \frac{7}{2}\right)$.



Method 2 (Equal radii)

We use the fact Q is of equal distance from each of the points A, B and C. In particular $QA^2 = QB^2 = QC^2$ or
$x^2 + y^2 = (x-2)^2 + (y-1)^2 = (x+3)^2 + (y-6)^2$
From the first equality,
$x^2 + y^2 = (x-2)^2 + (y-1)^2$
$4x + 2y = 5$



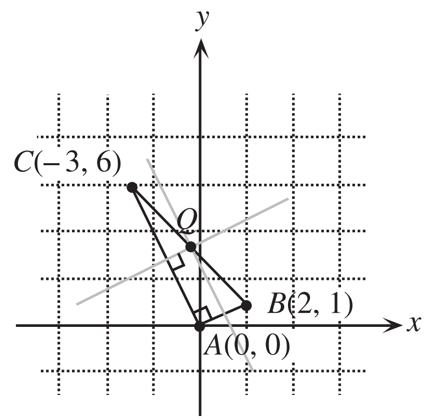
From the second equality,

$$(x-2)^2+(y-1)^2=(x+3)^2+(y-6)^2$$
$$-10x+10y=40$$
$$y=x+4$$

Substituting the equation above into into $4x+2y=5$, we obtain $4x+2(x+4)=5$ or $6x=-3$ or $x=-\frac{1}{2}$. Thus, $y=-\frac{1}{2}+4=\frac{7}{2}$, and so the centre of the circle is $\left(-\frac{1}{2},\frac{7}{2}\right)$.

Method 3 (Perpendicular bisectors)

We determine the equations for the perpendicular bisectors of *AB* and *AC*. The centre is the point of intersection of these two lines.



Since *AB* has slope $\frac{1}{2}$, then the slope of its perpendicular bisector is -2. Since the midpoint of *AB* is $\left(1,\frac{1}{2}\right)$, then the perpendicular bisector is $y-\frac{1}{2}=-2(x-1)$ or $y=-2x+\frac{5}{2}$.

Since *AC* has slope -2, then the slope of its perpendicular bisector is $\frac{1}{2}$. Since the midpoint of *AB* is $\left(-\frac{3}{2},3\right)$, then the perpendicular bisector is $y-3=\frac{1}{2}\left(x+\frac{3}{2}\right)$ or $y=\frac{1}{2}x+\frac{15}{4}$.

To find the point of intersection of these two lines, we set them equal:

$$-2x+\frac{5}{2}=\frac{1}{2}x+\frac{15}{4}$$
$$-\frac{5}{4}=\frac{5}{2}x$$
$$x=-\frac{1}{2}$$

From this, $y=-2\left(-\frac{1}{2}\right)+\frac{5}{2}=\frac{7}{2}$, and so the centre of the circle is $\left(-\frac{1}{2},\frac{7}{2}\right)$.

7. (a) *Solution 1*

Using a known formula for the area of a triangle, $A=\frac{1}{2}ab\sin C$,

$$18=\frac{1}{2}(2x+1)(2x)\sin 30^\circ$$
$$36=(2x+1)(2x)\left(\frac{1}{2}\right)$$
$$0=2x^2+x-36$$
$$0=(2x+9)(x-4)$$

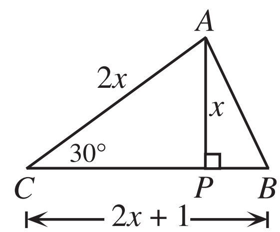
and so $x=4$ or $x=-\frac{9}{2}$. Since *x* is positive, then $x=4$.

Solution 2

Draw a perpendicular from A to P on BC.

Using ΔAPC, $AP = AC\sin 30^\circ = 2x\left(\frac{1}{2}\right) = x$.

Now AP is the height of ΔABC, so Area $= \frac{1}{2}(BC)(AP)$.



Then

$$
\begin{aligned}
18 &= \tfrac{1}{2}(2x+1)(x) \\
0 &= 2x^2 + x - 36 \\
0 &= (2x+9)(x-4)
\end{aligned}
$$

and so $x = 4$ or $x = -\frac{9}{2}$.

Since x is positive, then $x = 4$.

Answer: $x = 4$

(b) Let the length of the ladder be L.

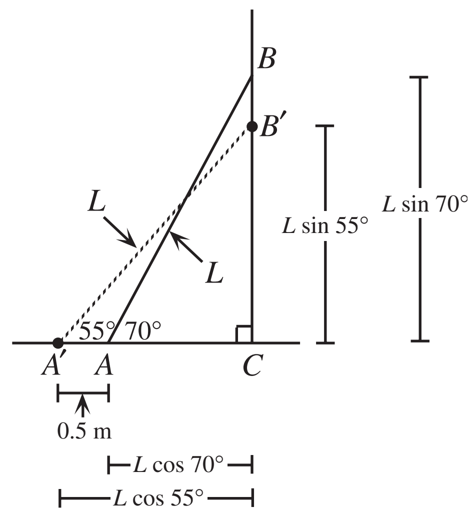
Then $AC = L\cos 70^\circ$ and $BC = L\sin 70^\circ$. Also,

$A'C = L\cos 55^\circ$ and $B'C = L\sin 55^\circ$.

Since $A'A = 0.5$, then

$0.5 = L\cos 55^\circ - L\cos 70^\circ$

$$L = \frac{0.5}{\cos 55^\circ - \cos 70^\circ} \qquad (*)$$



Therefore,

$$
\begin{aligned}
BB' &= BC - B'C \\
&= L\sin 70^\circ - L\sin 55^\circ \\
&= L\left(\sin 70^\circ - \sin 55^\circ\right) \\
&= \frac{(0.5)\left(\sin 70^\circ - \sin 55^\circ\right)}{\left(\cos 55^\circ - \cos 70^\circ\right)} \qquad \text{(from (*))} \\
&\approx 0.2603 \text{ m}
\end{aligned}
$$

Therefore, to the nearest centimetre, the distance that the ladder slides down the wall is 26 cm.

8. (a) *Solution 1*

In total, there are $\frac{1}{2}\times 5\times 20 = 50$ games played, since each of 5 teams plays 20 games (we divide by 2 since each game is double-counted).

In each game, there is either a loss or a tie.

The number of games with a loss is $44 + y$ from the second column, and the number of games with a tie is $\frac{1}{2}(11+z)$ (since any game ending in a tie has 2 ties).

So

$$\begin{aligned} 50 &= 44 + y + \tfrac{1}{2}(11 + z) \\ 100 &= 88 + 2y + 11 + z \\ 1 &= 2y + z \end{aligned}$$

Since y and z are non-negative integers, $z = 1$ and $y = 0$. So $x = 19$ since Team E plays 20 games.

Solution 2

In any game played, the final result is either both teams earning a tie, or one team earning a win, and the other getting a loss. Therefore, the total number of wins among all teams equals the total number of losses, ie.

$$25 + x = 44 + y$$

$$x - y = 19 \qquad (1)$$

Also, since team E plays 20 games, then

$$x + y + z = 20 \qquad (2)$$

So from (1), x must be at least 19, and from (2), x can be at most 20.
Lastly, we know that the total of all of the teams numbers of ties must be even, ie. $11 + z$ is even, ie. z is odd.
Since x is at least 19, then z can be at most 1 by (2).
Therefore, $z = 1$. Thus, $x = 19$ and $y = 0$.

Solution 3

In any game played, the final result is either both teams earning a tie, or one team earning a win, and the other getting a loss. Therefore, the total number of wins among all teams equals the total number of losses, ie.

$$25 + x = 44 + y$$

$$x - y = 19 \qquad (1)$$

Also, since team E plays 20 games, then

$$x + y + z = 20 \qquad (2)$$

So from (1), x must be at least 19, and from (2), x can be at most 20.
Consider the possibility that $x = 20$. From (2), then $y = z = 0$, which does not agree with (1).
Thus, the only possibility is $x = 19$. From (1), $y = 0$, and so $z = 1$ from (2). (These three values agree with both equations (1) and (2).)

(b) *Solution 1*

Assume such a sequence *a*, *b*, *c*, *d* exists. (We proceed by contradiction.)
Since the sum of any two consecutive terms is positive, $a+b>0$, $b+c>0$, and $c+d>0$. Adding these three inequalities, $(a+b)+(b+c)+(c+d)>0$ or $a+2b+2c+d>0$.
We are going to show that this statement contradicts the facts that are known about the sequence. We are told that the sum of any three consecutive terms is negative, ie. $a+b+c<0$ and $b+c+d<0$. Adding these two inequalities, $(a+b+c)+(b+c+d)<0$ or $a+2b+2c+d<0$.
This is a contradiction, since the two conditions $a+2b+2c+d>0$ and $a+2b+2c+d<0$ cannot occur simultaneously.
Therefore, our original assumption is false, and so no such sequence exists.

Solution 2

Assume such a sequence *a*, *b*, *c*, *d* exists. (We proceed by contradiction.)
We consider two cases.

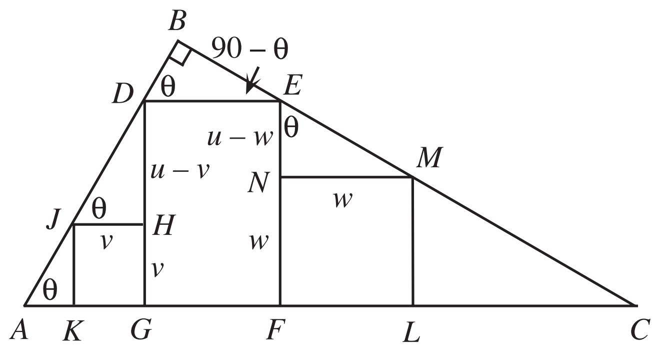
Case 1: $a \le 0$

In this case, $b>0$ since $a+b>0$.
Then, since $a+b+c<0$, we must have that $c<0$.
But $c+d>0$, so $d>0$.
This means that we have $b>0$ and $c+d>0$, ie. $b+c+d>0$.
But from the conditions on the sequence, $b+c+d<0$, a contradiction.
Therefore, no such sequence exists with $a \le 0$.

Case 2: $a>0$

In this case, it is not immediately clear whether *b* has to be positive or negative.
However, we do know that $a+b>0$ and $a+b+c<0$, so it must be true that $c<0$.
Then since $b+c>0$ and $c+d>0$, we must have both $b>0$ and $d>0$. But then $b+c+d=b+(c+d)>0$ since $c+d>0$ and $b>0$.
This is again a contradiction.
Therefore, no such sequence exists with $a>0$.

9. (a) Let $\angle BAC = \theta$. Then by parallel lines,
$\angle DJH = \angle BDE = \theta$.
Thus, $\angle BED = 90^\circ - \theta$ and so $\angle NEM = \theta$ since $\angle DEF = 90^\circ$.
Since $DG = u$ and $HG = v$, then $DH = u - v$.
Similarly, $EN = u - w$.



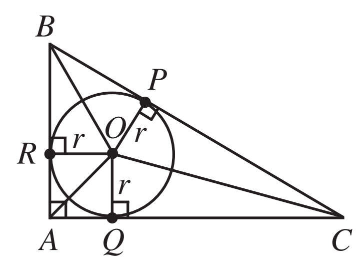
Looking at ΔDHJ and ΔMNE, we see that $\tan\theta = \dfrac{u-v}{v}$ and $\tan\theta = \dfrac{w}{u-w}$.

Therefore,

$$\frac{u-v}{v} = \frac{w}{u-w}$$
$$(u-v)(u-w) = vw$$
$$u^2 - uv - uw + vw = vw$$
$$u(u-v-w) = 0$$

and since $u \neq 0$, we must have $u - v - w = 0$ or $u = v + w$.
[Note: If $u = 0$, then the height of rectangle *DEFG* is 0, ie. *D* coincides with point *A* and *E* coincides with point *C*, which says that we must also have $v = w = 0$, ie. the squares have no place to go!]

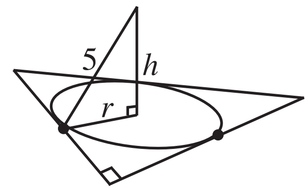
(b) Consider the cross-section of the sphere in the plane defined by the triangle. This cross-section will be a circle, since any cross-section of a sphere is a circle. This circle will be tangent to the three sides of the triangle, ie. will be the inscribed circle (or incircle) of the triangle. Let the centre of this circle be *O*, and its radius be *r*. We calculate the value of *r*.



Join *O* to the three points of tangency, *P*, *Q*, *R*, and to the three vertices *A*, *B*, *C*. Then *OP*, *OQ* and *OR* (radii) will form right angles with the three sides of the triangle. Consider the three triangles ΔAOB, ΔBOC and ΔCOA. Each of these triangles has a height of *r* and they have bases 15, 9 and 12, respectively. Since the area of ΔABC is equal to the sum of the areas of ΔAOB, ΔBOC, and ΔCOA,
So comparing areas,

$$\tfrac{1}{2}(9)(12)=\tfrac{1}{2}(9)(r)+\tfrac{1}{2}(12)(r)+\tfrac{1}{2}(15)(r)$$
$$54=\tfrac{1}{2}r(9+12+15)$$
$$r=3$$

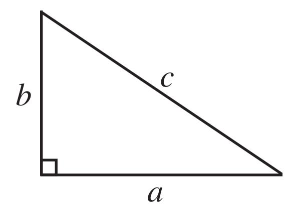
Now join the centre of the cross-sectional circle to the centre of the sphere and let this distance be h. Now, the line joining the centre of the circle to the centre of the sphere will be perpendicular to the plane of the triangle, so we can form a right-angled triangle by joining the centre of the sphere to any point on the circumference of the cross-sectional circle. By Pythagoras,



$$h^2+r^2=25$$
$$h=4$$

This tells us that the top of the sphere is 9 units above the plane of the triangle, since the top of the sphere is 5 units above the centre of the sphere.

10. (a) Consider a Pythagorean triangle with integer side lengths a, b, c satisfying $a^2+b^2=c^2$. To show that this triangle is Heronian, we must show that it has an integer area. Now we know that the area is equal to $\tfrac{1}{2}ab$, so we must show that either a or b is an even integer.



Suppose that both a and b are odd. (We proceed by contradiction.)
In this case, let $a=2k+1$ and $b=2l+1$. Then both a^2 and b^2 are odd, and so c^2 is even since $a^2+b^2=c^2$. Therefore, c itself must be even, so let $c=2m$.
Therefore,

$$(2k+1)^2+(2l+1)^2=(2m)^2$$
$$4k^2+4k+1+4l^2+4l+1=4m^2$$
$$4\left(k^2+k+l^2+l\right)+2=4\left(m^2\right)$$

But the right side is a multiple of 4, and the left side is not a multiple of 4. This is a contradiction.
Therefore, one of a or b must be even, and so the area of the triangle is an integer.
Thus, any Pythagorean triangle is Heronian.

(b) We examine the first few smallest Pythagorean triples:

3	4	5	$\left(3^2=4+5\right)$

5	12	13	$\left(5^2 = 12+13\right)$
6	8	10	(Does not fit pattern)
7	24	25	$\left(7^2 = 24+25\right)$

It appears from the first few examples that perhaps we can form a Pythagorean triple by using any odd number greater than 1 as its shortest leg.

Next, we notice from the pattern that the sum of the second leg and the hypotenuse is the square of the shortest leg, and that these two side lengths differ by 1.

Will this pattern always hold?

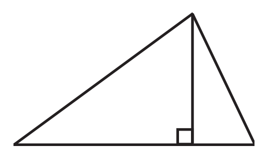
Let $a = 2k+1$ with $k \geq 1$. (This formula will generate all odd integers greater than or equal to 3.) Can we always find b so that $c = b+1$ and $a^2 + b^2 = c^2$?

Consider the equation

$$\begin{aligned}(2k+1)^2 + b^2 &= (b+1)^2 \\ 4k^2 + 4k + 1 + b^2 &= b^2 + 2b + 1 \\ 4k^2 + 4k &= 2b \\ b &= 2k^2 + 2k\end{aligned}$$

So we can always find a b to make the equation true. Therefore, since a can be any odd integer greater than or equal to 3, then we can make any odd number the shortest leg of a Pythagorean triangle, namely the Pythagorean triangle $a = 2k+1$, $b = 2k^2 + 2k$, $c = 2k^2 + 2k + 1$. (Check that $a^2 + b^2 = c^2$ does indeed hold here!)

(c) We consider forming a triangle by joining two Pythagorean triangles along a common side. Since any Pythagorean triangle is Heronian, then the triangle that is formed by joining two Pythagorean triangles in the manner shown will have integer side lengths and will have integer area, thus making it Heronian.



So again, we make a list of Pythagorean triples

3	4	5
5	12	13
6	8	10
7	24	25
8	15	17
9	40	41
10	24	26
11	60	61

We notice that we can scale any Pythagorean triangle by an integer factor and obtain another Pythagorean triangle. This will enable us to create two Pythagorean triangles with a common side length.
Also, we note that when joining two Pythagorean triangles, the hypotenuse of each triangle becomes a side length in the new triangle. Since we cannot have a side length divisible by 3, 5, 7 or 11, this eliminates the 3-4-5, 6-8-10, and 7-24-25 triangles from the list above.
Suppose we scale the 8-15-17 triangle by a factor of 4 to obtain 32-60-68 and join to the 11-60-61 triangle in the manner shown.
Thus we obtain a 43-61-68 triangle, which has integer area because its height is an even integer.
Therefore, a 43-61-68 triangle is Heronian.

[It is worth noting that this is not the *only* such triangle, but it is the one with the shortest sides.]