Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2003 Solutions
# Galois Contest (Grade 10)

1. (a) *Solution 1*

Since we are looking for 5 consecutive perfect squares which add to 1815, then the middle one of these squares should be close to $\frac{1}{5}(1815) = 363$.

What perfect square is closest to 363? Using a calculator, $\sqrt{363} \approx 19.05$, so $19^2 = 361$ is closest to 363.

So we try

$$17^2 + 18^2 + 19^2 + 20^2 + 21^2 = 289 + 324 + 361 + 400 + 441 = 1815$$

as we wanted.

So the largest of these integers is 21.

*Solution 2*

Let $n$ be the smallest of these 5 consecutive positive integers.

Then

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 &= 1815 \\ n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 + n^2 + 8n + 16 &= 1815 \\ 5n^2 + 20n + 30 &= 1815 \\ n^2 + 4n + 6 &= 363 \\ n^2 + 4n - 357 &= 0 \\ (n+21)(n-17) &= 0 \end{aligned}$$

Since $n$ is positive, then $n = 17$, and so the largest of the integers is $n + 4 = 21$.

*Solution 3*

Let $m$ be the middle of the 5 consecutive positive integers. (This will make the algebra easier.)

Then the 5 consecutive integers are $m-2$, $m-1$, $m$, $m+1$, and $m+2$, and so we have

$$\begin{aligned} (m-2)^2 + (m-1)^2 + m^2 + (m+1)^2 + (m+2)^2 &= 1815 \\ m^2 - 4m + 4 + m^2 - 2m + 1 + m^2 + m^2 + 2m + 1 + m^2 + 4m + 4 &= 1815 \\ 5m^2 + 10 &= 1815 \\ 5m^2 &= 1805 \\ m^2 &= 361 \\ m &= 19 \end{aligned}$$

since $m$ is positive. Therefore, the largest of the integers is $m + 2 = 21$.

(b) *Solution 1*

Let $n$ be the smallest of the 5 consecutive integers.

Then the sum of their squares is

$$n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2$$
$$=n^2+n^2+2n+1+n^2+4n+4+n^2+6n+9+n^2+8n+16$$
$$=5n^2+20n+30$$
$$=5(n^2+4n+6)$$

Since we have been able to factor out a 5 from the expression, then the sum of the squares of any 5 consecutive integers is divisible by 5.
(Note that $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2=5(n^2+4n+6)$ is divisible by 5 because the expression left in the parentheses after the 5 is factored out has to be an integer since $n$ is an integer.)

*Solution 2*
Let $m$ be the middle of the 5 consecutive integers. (This will make the algebra easier.) Then the 5 consecutive integers are $m-2$, $m-1$, $m$, $m+1$, and $m+2$, and so the sum of their squares is

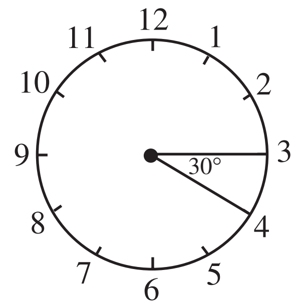
$$(m-2)^2+(m-1)^2+m^2+(m+1)^2+(m+2)^2$$
$$=m^2-4m+4+m^2-2m+1+m^2+m^2+2m+1+m^2+4m+4$$
$$=5m^2+10$$
$$=5(m^2+2)$$

which is divisible by 5, since we are able to factor out a 5.
(As before, note that $m^2+2$ is always an integer because $m$ itself is an integer.)

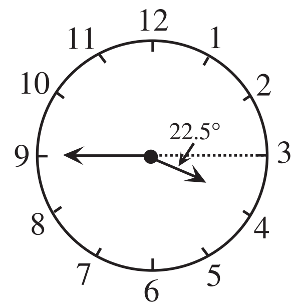
*Extension*
In (a), we saw that $17^2+18^2+19^2+20^2+21^2=1815$.
To express 1815 as the sum of 5 consecutive integers, we first figure out what the "middle" (or average) of the 5 integers is. The average is $\frac{1}{5}(1815)=363$, so
$361+362+363+364+365=1815$.
We want to determine the next integer larger than 1815 with this same property.
What is the next larger positive integer that is the sum of the squares of 5 consecutive integers?
In (b), we saw that if $m$ is the middle of 5 consecutive integers, then the sum of their squares is $(m-2)^2+(m-1)^2+m^2+(m+1)^2+(m+2)^2=5m^2+10$, and if $m=19$, then we get 1815 as the sum. The next larger value that is the sum of 5 squares will be when $m=20$, giving a sum of $5(20)^2+10=2010$.
Is 2010 the sum of 5 consecutive integers?
Here, the average number should be $\frac{1}{5}(2010)=402$, and in fact
$400+401+402+403+404=2010$.

Therefore, 2010 is the next integer larger than 1815 that is the sum of both 5 consecutive integers, and the sum of the squares of 5 consecutive integers.
(From what we have done here, it appears as if *any* multiple of 5 is the sum of 5 consecutive integers. Can you prove this?)

2. (a) Between 3:00 p.m. and 3:45 p.m., three-quarters of an hour has passed so the hour hand has moved three-quarters of the way from the "3" to the "4".
If we join each of the "3" and the "4" to the centre of the clock, the central angle of the sector formed is $\frac{1}{12}$ of the total way around the circle, or $30^\circ$.
Therefore, the angle through which the hour hand passes is $\frac{3}{4}\left(30^\circ\right) = 22.5^\circ$.




(b) *Solution 1*
If the hour hand was pointing exactly at the "3" at 3:45 p.m., the angle would be $180^\circ$, because the minute hand points exactly at the "9".
However, the hour hand has moved $22.5^\circ$ beyond the "3" (while the minute hand is pointing exactly at the "9"), so the angle between the hands is $180^\circ - 22.5^\circ = 157.5^\circ$.




*Solution 2*
For every hour that passes, the minute hand passes through $360^\circ$ and the hour hand passes through $30^\circ$. Therefore, in any one hour period, the minute hand passes through $330^\circ$ more than the hour hand. This implies that for every minute that passes the minute hand gains $5.5^\circ$ on the hour hand.
Between 3:00 p.m. and 3:45 p.m., the minute hand would gain $5.5^\circ \times 45 = 247.5^\circ$ on the hour hand.
Since the minute hand was $90^\circ$ behind the hour hand at 3:00 p.m., then at 3:45 p.m., the minute hand is $247.5^\circ - 90^\circ = 157.5^\circ$ ahead of the hour hand. This is the required angle.

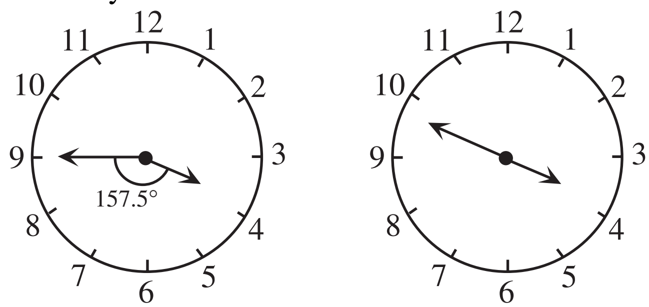
(c) *Solution 1*
At 3:45 p.m., the angle between the hands is $157.5^\circ$ and is *increasing*, because the minute hand moves faster than the hour hand, and the minute hand is "ahead" of the hour hand.
In one minute, the minute hand moves $\frac{1}{60}$ of the way around the clock, or $6^\circ$.
In one minute, the hour hand moves $\frac{1}{60}$ between two hour markings, or $\frac{1}{60}$ of $\frac{1}{12}$ of the total way around the clock, or $0.5^\circ$.

Therefore, the angle between the hands is increasing at a rate of $6^\circ - 0.5^\circ = 5.5^\circ$ per minute, since the hands are moving in the same direction.

From 3:45 p.m., the angle needs to increase by $180^\circ - 157.5^\circ = 22.5^\circ$ to reach $180^\circ$.

Therefore, it will take $\frac{22.5^\circ}{5.5^\circ} \approx 4.09$ minutes (or 4 minutes and 5 seconds) after 3:45 p.m. for the angle to become $180^\circ$.

Thus, the angle is $180^\circ$ at approximately 3:49 p.m.

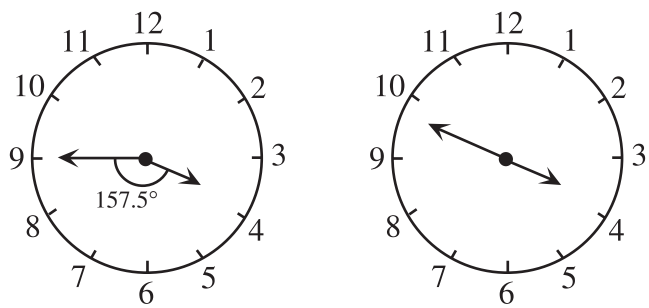



*Solution 2*

From Solution 2 in (b), we know that the minute hand gains $5.5^\circ$ every minute on the hour hand. At 3:00 p.m., the minute hand is $90^\circ$ behind the hour hand and we want it to be $180^\circ$ ahead of the hour hand.

In total, then, the minute hand must "make up" $270^\circ$ which will take $\frac{270^\circ}{5.5^\circ} = 49\frac{1}{11}$ minutes.

Thus, the angle is $180^\circ$ at approximately 3:49 p.m.

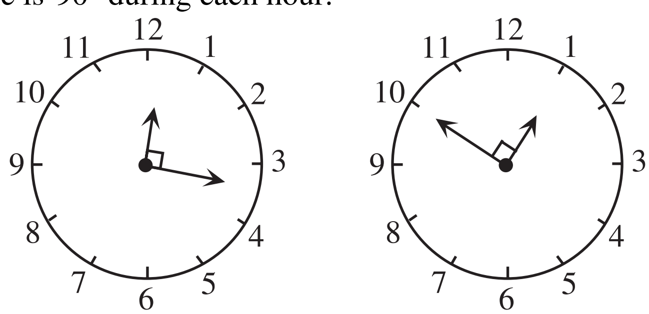



*Extension*

In any 12 hour period, every time between 12:00 and 11:59 will occur exactly once, so we can say that the 12 hour period begins at 12:00 and ends at 12:00.

We look for positions where the angle is $90^\circ$ during each hour.

Between 12:00 and 1:00, the hour hand will be between the 12 and 1. Thus, there will be one desired time when the minute hand is between the 3 and 4, and one where it is between the 9 and 10 (ie. one desired time between 12:15 and 12:20 and another between 12:45 and 12:50).



Between 1:00 and 2:00, the hour hand will be between the 1 and 2. Thus, there will be one desired time when the minute hand is between the 4 and 5, and one where it is between the 10 and 11.
We can continue with this reasoning for each of the 12 hours, and obtain 2 desired times for each hour slot. However, if one of the desired times occurs exactly on an hour, we will have counted it twice! (For instance, between 2:00 and 3:00 we will have counted one desired time when the minute hand is between the 11 and 12, while between 3:00 and 4:00 we will have counted one desired time when the minute hand is between the 12 and 1. These are in fact the same time – 3:00!)

At which hours is the angle between the hands equal to $90^\circ$? At 3:00 and 9:00. Therefore, we have double-counted two times, so the actually number of desired times is $2(12)-2=22$.

3. (a) We complete the chart, including at each step a description of the step.

This is one possible way for the game to be played in 8 moves.

| | | | | | |
|---|---|---|---|---|---|
| *Start* | *Q* | *Q* | | *D* | *D* |
| *Shift* | *Q* | | *Q* | *D* | *D* |
| *Jump* | *Q* | *D* | *Q* | | *D* |
| *Shift* | *Q* | *D* | *Q* | *D* | |
| *Jump* | *Q* | *D* | | *D* | *Q* |
| *Jump* | | *D* | *Q* | *D* | *Q* |
| *Shift* | *D* | | *Q* | *D* | *Q* |
| *Jump* | *D* | *D* | *Q* | | *Q* |
| *Shift* | *D* | *D* | | *Q* | *Q* |

There is in fact only one other possibility (which simply reverses whether a dime or quarter is being moved at each step).

(We can notice that, at each step, there is only one possible move that can be made to avoid having to backtrack.)

| | | | | | |
|---|---|---|---|---|---|
| *Start* | *Q* | *Q* | | *D* | *D* |
| *Shift* | *Q* | *Q* | *D* | | *D* |
| *Jump* | *Q* | | *D* | *Q* | *D* |
| *Shift* | | *Q* | *D* | *Q* | *D* |
| *Jump* | *D* | *Q* | | *Q* | *D* |
| *Jump* | *D* | *Q* | *D* | *Q* | |
| *Shift* | *D* | *Q* | *D* | | *Q* |
| *Jump* | *D* | | *D* | *Q* | *Q* |
| *Shift* | *D* | *D* | | *Q* | *Q* |

(b) In a game starting with 3 quarters and 3 dimes, the game board will have 7 squares. Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the 3 quarters to switch their order (ie. we are not allowed to jump one quarter over a second quarter).
Thus, the quarter that starts in the third square ends in the seventh square, the quarter that starts in the second square ends in the sixth square, and the quarter that starts in the first square ends in the fifth square.
Each quarter then moves 4 spaces, making a total of 12 squares moved. Similarly, each dime moves 4 spaces, making a total of 12 squares moved. In total, the coins move 24 squares. If this was done using only shifts, this would require 24 moves.

However, the game cannot be played in this fashion, because it is necessary to make jumps. Since we want the dimes and quarters to change positions, each dime needs to jump over (or be jumped over by) each of the 3 quarters. In other words, there need to 9 jumps made. Since each jump results in a move of 2 spaces, this "saves" 9 shifts.
Therefore, the number of required moves is at least $24-9=15$.
We have assumed here that no "backtracking" is done, so the game cannot be played in fewer than 15 moves.
(Can you construct the diagram to show how the game can be played in 15 moves?)

*Extension*

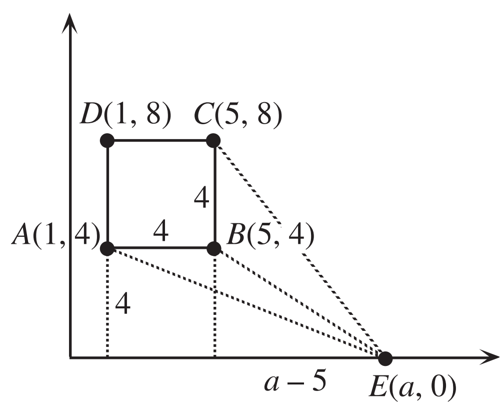
We use the same strategy as in (b).
In a game starting with $n$ quarters and $n$ dimes, the game board will have $2n+1$ slots.

Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the $n$ quarters to switch their order (ie. we are not allowed to jump on quarter over another).
Thus, the quarter that starts in the first square ends in the $(n+2)$th square, the quarter that starts in the second square ends in the $(n+3)$th square, and so on, with the quarter that starts in the $n$th square ending in the $(2n+1)$th square.
Each of the $n$ quarters has moved a total of $n+1$ squares, so the quarters have therefore moved a total of $n(n+1)$ squares.
Similarly, the dimes have moved a total $n(n+1)$ squares.
So the coins have moved a total of $2n(n+1)=2n^2+2n$ squares. If this was done using only shifts, this would require $2n^2+2n$ moves.
Using exactly the same reasoning as in (c), the required number of moves is
$2n^2+2n-n^2=n^2+2n=n(n+2)$.
(A question that should be asked is "Can the game be played in exactly $n(n+2)$ moves?" In order to answer this question, we would need to come up with a general strategy that would allow us to play the game in this number of moves, no matter what value $n$ takes.)

4. (a) Since $ABCD$ is a square and $AD$ has side length 4, then each of the sides of $ABCD$ has length 4. We can also conclude that $B$ has coordinates $(5,4)$ and $C$ has coordinates $(5,8)$. (Since $AD$ is parallel to the $y$-axis, then $AB$ is parallel to the $x$-axis.) The area of square $ABCD$ is thus $4^2=16$.

Since the $\Delta CBE$ and $\Delta ABE$ lie entirely outside square $ABCD$ then $E$ must lie below the line $AB$ (which it does), and to the right of the line $CB$, so $a>5$.

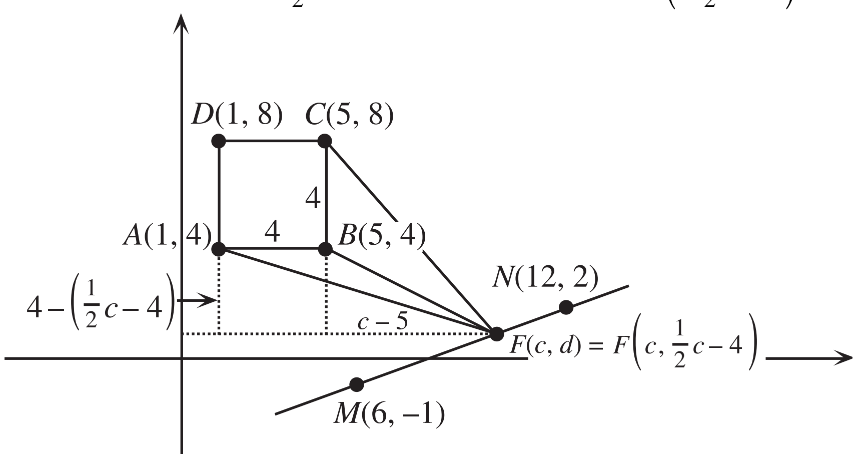
First, we consider $\Delta ABE$. We can think of $AB$, which has length 4, as being the base of $\Delta ABE$. The length of the height of $\Delta ABE$ is the distance from $E$ to the line through $A$ and $B$. This distance is 4, since $AB$ is parallel to the $x$-axis. Thus, the area of $\Delta ABE$ is $\frac{1}{2}bh=\frac{1}{2}(4)(4)=8$. (Note that the area of $\Delta ABE$ is *always* 8, provided that $a>5$.)




Next, we look at $\Delta CBE$. We can think of $CB$, which has length 4, as being the base of $\Delta CBE$. The length of the height of $\Delta CBE$ is the distance from $E$ to the line through $C$ and $B$. This distance is $a-5$, since $BC$ is parallel to the $y$-axis. Therefore, the area of $\Delta CBE$ is $\frac{1}{2}bh=\frac{1}{2}(4)(a-5)=2a-10$.

Since we would like the sum of the areas of these two triangles to be equal to the area of the square, then $8+(2a-10)=16$ or $a=9$.
Therefore, $a=9$.
[Note that this can also be done by subtraction of areas.]

(b) Let the point $F$ have coordinates $(c,d)$.
Since triangle $CBF$ lies entirely outside the square, $F$ lies to the right of the square, ie. $c>5$. Since triangle $ABF$ lies entirely outside the square, $F$ lies below the square, ie. $d<4$.
Now $F$ lies on the line passing through both $M$ and $N$. What is the equation of this line?
The slope is $\frac{2-(-1)}{12-6}=\frac{1}{2}$, so the equation of the line is $y-2=\frac{1}{2}(x-12)$ or $y=\frac{1}{2}x-4$.
Since $F$ lies on this line, then $d=\frac{1}{2}c-4$, so $F$ has coordinates $\left(c,\frac{1}{2}c-4\right)$.



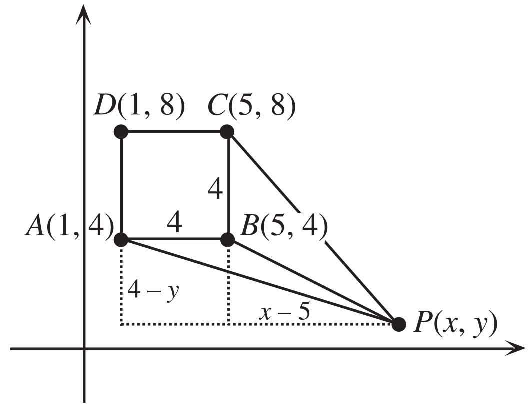

We consider next $\Delta ABF$. This triangle has base $AB$ of length 4. The height of $\Delta ABF$ is the vertical distance from $F$ to the line through $A$ and $B$, which is $4-\left(\frac{1}{2}c-4\right)=8-\frac{1}{2}c$.
Therefore, the area of $\Delta ABF$ is $\frac{1}{2}bh=\frac{1}{2}(4)\left(8-\frac{1}{2}c\right)=16-c$.

$\Delta CBF$ has base $CB$ of length 4. The height of $\Delta CBF$ is the horizontal distance from $F$ to the line through $C$ and $B$, which is $c-5$. Therefore, the area of $\Delta CBF$ is $\frac{1}{2}bh=\frac{1}{2}(4)(c-5)=2c-10$.

Thus, we are looking for the value of $c$ for which $16-c+2c-10=16$, and so $c=10$.
Therefore, $F$ is the point $(10,1)$.

*Extension*

Since triangle *CBP* lies entirely outside the square, *P* lies to the right of the square, ie. $x > 5$. Since triangle *ABP* lies entirely outside the square, *P* lies below the square, ie. $y < 4$.




We consider next $\Delta ABP$. This triangle has base *AB* of length 4. The height of $\Delta ABP$ is the vertical distance from *P* to the line through *A* and *B*, which is $4 - y$. Therefore, the area of $\Delta ABP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(4-y) = 8 - 2y$.

$\Delta CBP$ has base *CB* of length 4. The height of $\Delta CBP$ is the horizontal distance from *P* to the line through *C* and *B*, which is $x - 5$. Therefore, the area of $\Delta CBP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(x-5) = 2x - 10$.

Since the sum of the areas of $\Delta ABP$ and $\Delta CBP$ is equal to the area of the square, then

$$\begin{aligned}(8-2y)+(2x-10) &= 16\\ 2x-18 &= 2y\\ y &= x-9\end{aligned}$$

Therefore, the points *P* which satisfy the condition that the sum of the areas of $\Delta ABP$ and $\Delta CBP$ is equal to the area of the rectangle are all of the points on the line $y = x - 9$ with $x > 5$ and $y < 4$.

[Note that we could include the two endpoints $(5,-4)$ and $(13,4)$ if we allowed the area of one of the two triangles to be equal to 0.]