Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Fermat Contest (Grade 11)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

2004 Fermat Contest Solutions

1. Calculating,
$$\frac{10}{10(11)-10^2}=\frac{10}{110-100}=1$$
Answer: (D)

2. $\sqrt{4^0+4^2+4^3}=\sqrt{1+16+64}=\sqrt{81}=9$
Answer: (A)

3. First, we determine that $x=4$ and $y=3$. Therefore, $x-y=1$.
Answer: (B)

4. Since the loaf is cut into 25 pieces of equal volume, the volume of each piece is
$$\frac{(20\text{ cm})\times(18\text{ cm})\times(5\text{ cm})}{25}=(4\text{ cm})\times(18\text{ cm})\times(1\text{ cm})=72\text{ cm}^3.$$
Since the density of the loaf is $2\text{ g}/\text{cm}^3$, then the mass of each piece is volume times density, or $\left(2\text{ g}/\text{cm}^3\right)\times\left(72\text{ cm}^3\right)=144\text{ g}$.
Answer: (D)

5. *Solution 1*
Taking reciprocals of both sides,
$$\begin{aligned}\left(\frac{1}{2+3}\right)\left(\frac{1}{3+4}\right)&=\frac{1}{x+5}\\(2+3)(3+4)&=x+5\\35&=x+5\\x&=30\end{aligned}$$

Solution 2
Simplifying the left side,
$$\begin{aligned}\left(\frac{1}{2+3}\right)\left(\frac{1}{3+4}\right)&=\frac{1}{x+5}\\\left(\frac{1}{5}\right)\left(\frac{1}{7}\right)&=\frac{1}{x+5}\\\frac{1}{35}&=\frac{1}{x+5}\end{aligned}$$
Comparing denominators gives $x+5=35$ or $x=30$.
Answer: (C)

6. If three cans fill $\frac{2}{3}$ L, then one can will fill $\frac{2}{9}$ L.

   Therefore, the total number of cans required to fill 8 L is $\dfrac{8}{\left(\frac{2}{9}\right)} = 8 \times \dfrac{9}{2} = 36.$

   Answer: (A)

7. *Solution 1*
   We first simplify the expression:
   $$\frac{x^2-4}{x^2-2x} = \frac{(x+2)(x-2)}{x(x-2)} = \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x}$$
   (We can cancel the factor of $x-2$ since x is not equal to 2.)

   Substituting $x = \frac{1}{5}$, we get $1 + \dfrac{2}{\left(\frac{1}{5}\right)} = 1 + 10 = 11.$

   Solution 2
   Substituting $x = \frac{1}{5}$,
   $$\frac{x^2-4}{x^2-2x} = \frac{\frac{1}{25}-4}{\frac{1}{25}-\frac{2}{5}} = \frac{\frac{1}{25}-\frac{100}{25}}{\frac{1}{25}-\frac{10}{25}} = \frac{-\frac{99}{25}}{-\frac{9}{25}} = 11.$$

   Answer: (E)

8. From the graph, Jane arrives with 10 L of gas in her gas tank, and leaves with 50 L of gas, having paid \$36.60. So she buys 40 L of gas for \$36.60, so the cost per litre is $\dfrac{\$36.60}{40} = \$0.915$, or 91.5 cents.

   Answer: (A)

9. 4% of 10 000 is $\frac{4}{100} \times 10000 = 400$, so the population of Cayleyville in 2004 is 10 400.
   12% of 25 000 is $\frac{12}{100} \times 25000 = 3000$, so the population of Pascalberg in 2004 is 22 000.
   So the population difference in 2004 is $22000 - 10400 = 11600$.

   Answer: (B)

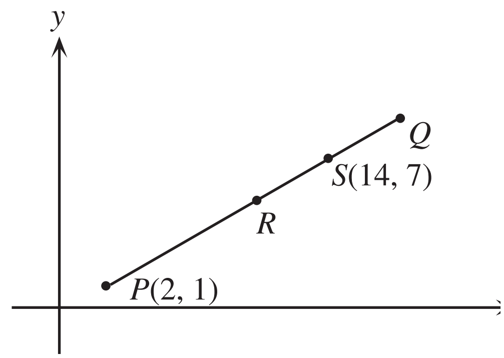
10. From the two given balances, 3 ▲'s balance 5 ●'s, and 1 ▲ balances 2 ■'s and 1 ●.
    Tripling the quantities on the second balance implies that 3 ▲'s will balance 6 ■'s and 3 ●'s.
    Therefore, 5 ●'s will balance 6 ■'s and 3 ●'s, and removing 3 ●'s from each side implies that 2 ●'s will balance 6 ■'s, or 1 ● will balance 3 ■'s.

    Answer: (C)

11. Since x is between –1 and 0, then x^2 is between 0 and 1, and so $-x^2$ is between –1 and 0. Therefore, the best letter is either *b* or *c*. When a number between –1 and 0 is squared, it becomes closer to 0 than it was before, so the best choice must be *c*, not *b*.

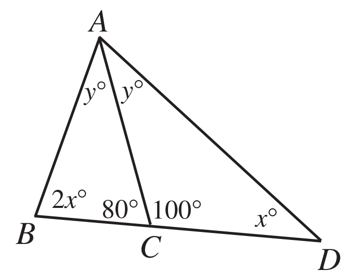
    Answer: (C)

12. Since R is the midpoint of PQ and S is the midpoint of QR, then S is $\frac{3}{4}$ of the way from P to Q.
Since S is 12 units to the right of P, then Q is $\frac{4}{3}\times 12 = 16$ units to the right of P.
Since S is 6 units up from P, then Q is $\frac{4}{3}\times 6 = 8$ units up from P.
Therefore, the coordinates of S are $(2+16,\ 1+8) = (18,9)$.



Answer: (D)

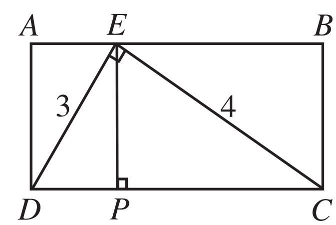
13. In triangle ACD, $x^\circ + y^\circ + 100^\circ = 180^\circ$, so $x + y = 80$ (*).
Since $\angle ACB$ and $\angle ACD$ are supplementary, then $\angle ACB = 180^\circ - \angle ACD = 80^\circ$.
Thus, in triangle ACB, $2x^\circ + y^\circ + 80^\circ = 180^\circ$, so $2x + y = 100$ (**).
Subtracting (*) from (**), we obtain $x = 20$.



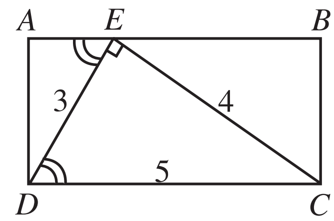
Answer: (E)

14. *Solution 1*
By the Pythagorean Theorem, $DC^2 = DE^2 + EC^2$ so $DC = 5$.
Draw a line from E to point P on DC so that EP is perpendicular to DC. Since $ABCD$ is a rectangle, then $AD = EP$.
Then the area of triangle DEC is equal to $\frac{1}{2}(DE)(EC)$ and also to $\frac{1}{2}(DC)(EP)$.
So

$$\frac{1}{2}(3)(4) = \frac{1}{2}(5)(EP)$$
$$12 = 5(AD)$$
$$AD = \frac{12}{5} = 2.4$$



Solution 2



By the Pythagorean Theorem, $DC^2 = DE^2 + EC^2$
so $DC = 5$.

Then $\sin(\angle EDC) = \dfrac{EC}{DC} = \dfrac{4}{5}$.

But $\angle AED = \angle EDC$ since AB and DC are parallel,

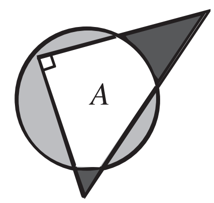
so $\dfrac{4}{5} = \sin(\angle AED) = \dfrac{AD}{ED} = \dfrac{AD}{3}$, and so $AD = \dfrac{12}{5} = 2.4$.

Answer: (B)

15. If $x^2 - y^2 = 0$, then $(x-y)(x+y) = 0$, so $y = x$ or $y = -x$. These are the equations of two straight lines (each of which passes through $(0,0)$).

Answer: (E)

16. Let the area that is inside the triangle and inside the circle be A, and let the area outside the triangle but inside the circle be B. Then B is also equal to the area outside the circle but inside the triangle.



We can then see that $A + B$ is equal to the area of the circle and is also equal to the area of the
triangle, so the circle and triangle have the same area.

Thus, if r is the radius of the circle, then $\pi r^2 = \frac{1}{2}(6)(8)$ or $\pi r^2 = 24$ or $r = \sqrt{\dfrac{24}{\pi}} \approx 2.8$.

Answer: (B)

17. Since the difference between consecutive terms is constant, then the difference between the third and fourth terms is equal to the difference between the first and second terms, or

$$\begin{aligned}(x + 2y + 2) - (3x + y) &= y - x\\ y - 2x + 2 &= y - x\\ 2 &= x\end{aligned}$$

Thus, we can rewrite the sequence as 2, y, $y + 6$, and $2y + 4$.

Again, since the difference is constant and the difference between the second and third terms is 6, then the difference between the first and second terms is 6, ie. $y = 8$.

Thus, $y - x = 6$.

Answer: (E)

18. *Solution 1*

We expand the two expressions.

First, $y = a(x-2)^2 + c = a(x^2 - 4x + 4) + c = ax^2 - 4ax + (4a + c)$.

Second, $y = (2x-5)(x-b) = 2x^2 - (5+2b)x + 5b$.

We can then equate the coefficients.

From the leading coefficients, $a = 2$.

From the coefficients of x, $4a = 8 = 5 + 2b$ or $b = \frac{3}{2}$.

Solution 2

The x-coordinate of the vertex of a parabola is the average value of the roots of the parabola.
From the first of the given forms, the vertex has x coordinate 2.
From the second of the given forms, the roots are $x = \frac{5}{2}$ and $x = b$.
Therefore, $\frac{1}{2}\left(\frac{5}{2} + b\right) = 2$ or $\frac{5}{2} + b = 4$ or $b = \frac{3}{2}$.

Answer: (B)

19. Let P be the original price the retailer sets.
Then one-half of the initial 1200, or 600 copies, will sell for a price of P, giving revenue of $600P$.
Two-thirds of the remaining 600, or 400 copies, will sell for a price of $0.6P$ (ie. 40% off of P), giving revenue of $400(0.6P) = 240P$.
The remaining 200 copies will sell for a price of $0.25P$ (ie. 75% off of P), giving revenue of $200(0.25P) = 50P$.
To make a reasonable profit, her revenue must be \$72 000, or

$$\begin{aligned} 600P + 240P + 50P &= 72000 \\ 890P &= 72000 \\ P &\approx 80.90 \end{aligned}$$

Thus, she should set an original price of \$80.90.

Answer: (D)

20. The ball is rolling towards Marcos at 4 m/s and he is running towards it at 8 m/s, so he gains 12 metres per second on the ball. Since he starts 30 m from the ball, it will take him $\frac{30}{12} = 2.5$ s to reach the ball.
The ball is rolling away from Michael at 4 m/s and he is running at 9 m/s, so he is gaining 5 m/s on the ball. Since he starts 15 m behind the ball, he would catch up to the ball in 3 s if it continued to roll.
Thus, Marcos gets to the ball first. After 2.5 s, the Michael has gained $5(2.5) = 12.5$ m on the ball, so is 2.5 m from the ball when Marcos touches it first.

Answer: (C)

21. *Solution 1*

In one hour, Bill paints $\dfrac{1}{B}$ of the line and Jill paints $\dfrac{1}{J}$ of the line.
Let t be number of hours during which both Bill and Jill paint the line. Since Bill paints for one hour before Jill starts to paint and since the painting of the line is completely finished after the t hours that they both work, then $\dfrac{1}{B} + t\left(\dfrac{1}{B} + \dfrac{1}{J}\right) = 1$ or

$$t = \frac{1 - \frac{1}{B}}{\frac{1}{B} + \frac{1}{J}} = \frac{\left[\frac{B-1}{B}\right]}{\left[\frac{B+J}{BJ}\right]} = \frac{J(B-1)}{B+J}.$$

So Bill works for $t+1=\dfrac{J(B-1)}{B+J}+1=\dfrac{BJ-J}{B+J}+\dfrac{B+J}{B+J}=\dfrac{BJ+B}{B+J}=\dfrac{B(J+1)}{B+J}$ hours.

Solution 2
Suppose that Bill could paint the line in 1 hour only, ie. $B=1$.
Then when Jill joins the painting after 1 hour, Bill would actually be finished the painting. In other words, if $B=1$, then the total time that Bill spends painting the line, regardless of the value of *J*, is 1 hour. If we substitute $B=1$ into the five choices, we obtain

(A) $\dfrac{J+1}{J+1}=1$ (B) $J+1$ (C) $\dfrac{J}{J+1}+1$
(D) $\dfrac{J}{2}$ (E) $\dfrac{J-1}{J+1}$

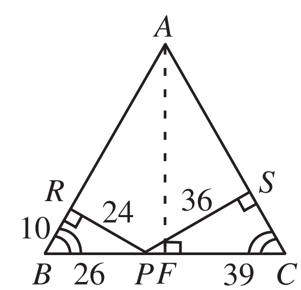
The only choice which is equal to 1, regardless of the value of *J*, is the first one.

Answer: (A)

22. Since we would like the product to have 303 digits, then we would like this product to be greater than 10^{302} but less than 10^{303}.
We will start by trying $k=300$. In this case, $\left(2^k\right)\left(5^{300}\right)=\left(2^{300}\right)\left(5^{300}\right)=10^{300}$, so we want *k* to be bigger than 300.
Each time we increase *k* by 1, the existing product is multiplied by 2. For the final product to have 303 digits, we need to multiply 10^{300} by a power of 2 between 100 and 1000. The smallest power of 2 that satisfies this is $2^7=128$.
Therefore, we would like $k=307$. In this case,
$$\left(2^{307}\right)\left(5^{300}\right)=\left(2^7\right)\left(2^{300}\right)\left(5^{300}\right)=\left(2^7\right)\left(10^{300}\right)=128\times10^{300}.$$
When this number is expanded, the digits are 128 followed by 300 zeros. Therefore, the sum of the digits is 11.

Answer: (A)

23. Since triangle *ABC* is isosceles, $\angle ABC=\angle ACB$, and so triangle *BRP* is similar to triangle *CSP* (equal angle, right angle).



Therefore, $\dfrac{BP}{RP}=\dfrac{CP}{SP}$ or $\dfrac{BP}{24}=\dfrac{CP}{36}$ or $\dfrac{BP}{CP}=\dfrac{2}{3}$.
But since *BC* has length 65 cm, then $BP+CP=65$ cm, and so $BP=26$ cm and $BP=39$ cm. Since $BP=26$ cm and $RP=24$ cm, then by the Pythagorean Theorem, $BR=10$ cm.
Next, we drop a perpendicular from *A* to *F* on *BC*. Since triangle *ABC* is isosceles, $BF=\frac{1}{2}(BC)=\frac{65}{2}$ cm.
Also, triangle *BFA* is similar to triangle *BRP* (common angle, right angle).
Thus, $\dfrac{BR}{RP}=\dfrac{BF}{FA}$, so $FA=\dfrac{\left(\frac{65}{2}\right)(24)}{10}=78$ cm.
The area of triangle *ABC* is then $\frac{1}{2}(BC)(FA)=\frac{1}{2}(65)(78)=2535\text{ cm}^2$.

Answer: (D)

24. Since the difference between $f(x)$ and $f(x-2)$ has degree 2, then the degree of $f(x)$ is at least 2. If the degree of $f(x)$ was equal to 2, then the x^2 terms of $f(x)$ and $f(x-2)$ would cancel when we subtracted. (Try this.) Therefore, the degree of $f(x)$ must be at least 3.

So we try $f(x) = ax^3 + px^2 + qx + r$.

Thus,

$$\begin{aligned} f(x-2) &= a(x-2)^3 + p(x-2)^2 + q(x-2) + r \\ &= a\left(x^3 - 6x^2 + 12x - 8\right) + p\left(x^2 - 4x + 4\right) + q(x-2) + r \\ &= ax^3 + (-6a+p)x^2 + (12a - 4p + q)x + (-8a + 4p - 2q + r) \end{aligned}$$

We are told that $f(x) - f(x-2) = (2x-1)^2$, so

$$\left[ax^3 + px^2 + qx + r\right] - \left[ax^3 + (-6a+p)x^2 + (12a-4p+q)x + (-8a+4p-2q+r)\right] = 4x^2 - 4x + 1$$

$$6ax^2 + (-12a + 4p)x + (8a - 4p + 2q) = 4x^2 - 4x + 1$$

Comparing coefficients,

$$\begin{aligned} 6a &= 4 \\ -12a + 4p &= -4 \\ 8a - 4p + 2q &= 1 \end{aligned}$$

From the first equation, $a = \frac{2}{3}$.

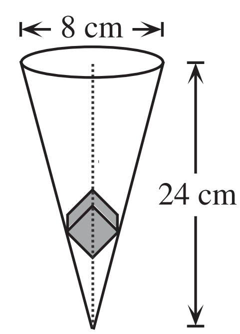
Substituting $6a = 4$ into the second equation, we obtain $-8 + 4p = -4$ or $p = 1$.

Substituting these two values into the third equation, we obtain $8\left(\frac{2}{3}\right) - 4(1) + 2q = 1$ or $q = -\frac{1}{6}$.

Therefore, $p + q = 1 + \left(-\frac{1}{6}\right) = \frac{5}{6}$.

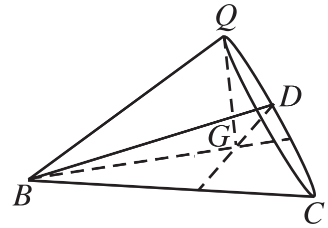
Answer: (B)

25. The first difficult thing about this problem is visualizing the position of the cube inside the cone. From there, we must determine how to calculate the required distance. Since the cube is balanced in an upright position with the axis of the cone coincides with one of the internal diagonals of the cube, then the cube is oriented with one vertex, *A*, pointing downwards, three vertices, *B*, *C* and *D*, touching the walls of the cone, and one vertex, *Q*, pointing upwards.



(We do not need to consider the remaining three vertices.) By symmetry, *B*, *C* and *D* lie in a plane parallel to the base of the cone and form an equilateral triangle.

We draw a diagram to illustrate the configuration of B, C, D, and Q. In the diagram, QB is the length of the diagonal of the face of a cube (this requires careful visualization of where the faces of the cube are positioned) and G is the intersection of the three medians. By symmetry, vertex A will lie directly beneath G in a line from Q. Thus, the line QGA lies along the axis of the cone.



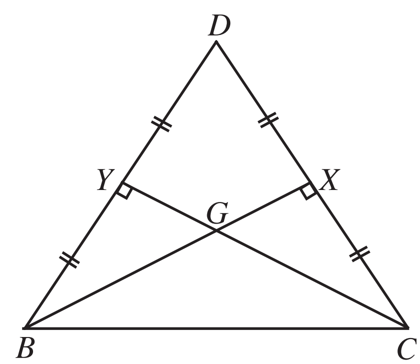
We can now determine the required distance by determining

- the vertical distance from the tip of the cone, T, at which B, C and D touch the cone, and
- the distance between A and the plane formed by B, C and D,

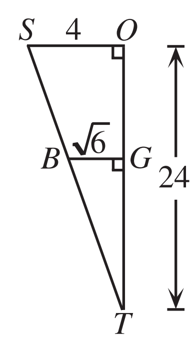
and then subtracting these two distances.

Now for some calculations.
Since each edge of the tetrahedron is a diagonal of a face of the cube, then each has length $\sqrt{3^2+3^2}=3\sqrt{2}$. In particular, BQ, BC, BD and CD all have length $3\sqrt{2}$.

Next, if we extract triangle BCD, then and draw the medians BX and CY which intersect at G, then we see that triangle BGY is a 30-60-90 triangle, so $BG=\frac{2}{\sqrt{3}}(BY)=\frac{2}{\sqrt{3}}\left(\frac{1}{2}(BA)\right)=\frac{1}{\sqrt{3}}\left(3\sqrt{2}\right)=\sqrt{6}$. This tells us that the distance from the axis of the cone to the points where the cube touches the cone is $\sqrt{6}$.



We can now calculate the distance from T to the plane containing B, C and D. To do, this we look at a partial cross-section of the cone labelling B, T, G, as well as O, the centre of the base of the cone, and S, the remaining vertex of the cross-section.
We can see that triangle TGB is similar to triangle TOS, so $\frac{TG}{TO}=\frac{GB}{OS}$ or $TG=24\left(\frac{\sqrt{6}}{4}\right)=6\sqrt{6}$,
(since the radius of the base of the cone is 4).



Lastly, we need to calculate the distance AG. To do this, we can calculate AQ and QG and subtract these two distances.

AQ is the main diagonal of the cube, so $AQ = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$.

Looking at triangle BGQ, which is right-angled, we see that

$QG = \sqrt{BQ^2 - BG^2} = \sqrt{\left(3\sqrt{2}\right)^2 - \left(\sqrt{6}\right)^2} = 2\sqrt{3}$.

Therefore, $AG = AQ - QG = \sqrt{3}$.

Therefore, the required distance is $TA = TG - AG = 6\sqrt{6} - \sqrt{3}$.

Answer: (A)