2008 Galois Contest (Grade 10)

Wednesday, April 16, 2008

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1. Three positive integers a, b and x form an O'Hara triple (a, b, x) if $\sqrt{a} + \sqrt{b} = x$. For example, $(1, 4, 3)$ is an O'Hara triple because $\sqrt{1} + \sqrt{4} = 3$.

   (a) If $(36, 25, x)$ is an O'Hara triple, determine the value of x.

   (b) If $(a, 9, 5)$ is an O'Hara triple, determine the value of a.

   (c) Determine the five O'Hara triples with $x = 6$. Explain how you found these triples.

2. (a) Determine the equation of the line passing through the points $P(0, 5)$ and $Q(6, 9)$.

   (b) A line, through Q, is perpendicular to PQ. Determine the equation of the line.

   (c) The line from (b) crosses the x-axis at R. Determine the coordinates of R.

   (d) Determine the area of right-angled $\triangle PQR$.

3. (a) A class of 20 students was given a two question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	18
2	14

   Determine the smallest possible number and the largest possible number of students that could have answered both questions correctly. Explain why these are the smallest and largest possible numbers.

   (b) A class of 20 students was given a three question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	18
2	14
3	12

   Determine the smallest possible number and the largest possible number of students that could have answered all three questions correctly. Explain why these are the smallest and largest possible numbers.

   (c) A class of 20 students was given a three question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	x
2	y
3	z

   where $x \geq y \geq z$ and $x + y + z \geq 40$.
   Determine the smallest possible number of students who could have answered all three questions correctly in terms of x, y and z.

4. Carolyn and Paul are playing a game starting with a list of the integers 1 to n. The rules of the game are:

- Carolyn always has the first turn.
- Carolyn and Paul alternate turns.
- On each of her turns, Carolyn must remove one number from the list such that this number has at least one positive divisor other than itself remaining in the list.
- On each of his turns, Paul must remove from the list all of the positive divisors of the number that Carolyn has just removed.
- If Carolyn cannot remove any more numbers, then Paul removes the rest of the numbers.

For example, if $n = 6$, a possible sequence of moves is shown in this chart:

Player	Number(s) removed	Number(s) remaining	Notes
Carolyn	4	1, 2, 3, 5, 6	
Paul	1, 2	3, 5, 6	
Carolyn	6	3, 5	She could not remove 3 or 5
Paul	3	5	
Carolyn	None	5	Carolyn cannot remove any number
Paul	5	None	

In this example, the sum of the numbers removed by Carolyn is $4 + 6 = 10$ and the sum of the numbers removed by Paul is $1 + 2 + 3 + 5 = 11$.

(a) Suppose that $n = 6$ and Carolyn removes the integer 2 on her first turn. Determine the sum of the numbers that Carolyn removes and the sum of the numbers that Paul removes.

(b) If $n = 10$, determine Carolyn's maximum possible final sum. Prove that this sum is her maximum possible sum.

(c) If $n = 14$, prove that Carolyn cannot remove 7 numbers.