The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Fermat Contest

(Grade 11)

Thursday, February 21, 2013
(in North America and South America)

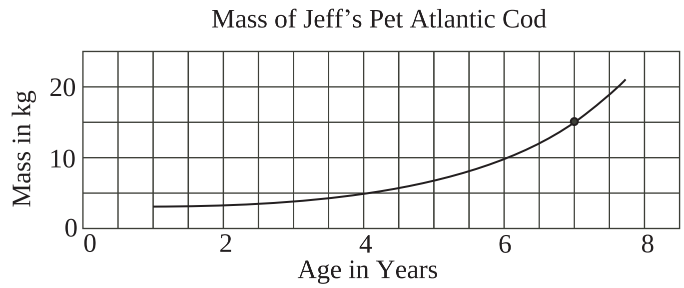
Friday, February 22, 2013
(outside of North America and South America)

Solutions

1. Simplifying, $\dfrac{10^2+6^2}{2} = \dfrac{100+36}{2} = \dfrac{136}{2} = 68$.

   ANSWER: (D)

2. A mass of 15 kg is halfway between 10 kg and 20 kg on the vertical axis.
   The point where the graph reaches 15 kg is halfway between 6 and 8 on the horizontal axis.

   Therefore, the cod is 7 years old when its mass is 15 kg.

   ANSWER: (B)

3. Each interior angle in a square is 90°. In particular, $\angle SPQ = 90^\circ$.
   Each interior angle in an equilateral triangle is 60°. In particular, $\angle TPQ = 60^\circ$.
   PR is a diagonal of square $PQRS$. Thus, it bisects angle $\angle SPQ$, with $\angle SPR = \angle RPQ = 45^\circ$.
   Therefore, $\angle TPR = \angle TPQ + \angle QPR = 60^\circ + 45^\circ = 105^\circ$.

   ANSWER: (B)

4. Since the tick marks divide the cylinder into four parts of equal volume, then the level of the milk shown is a bit less than $\frac{3}{4}$ of the total volume of the cylinder.
   Three-quarters of the total volume of the cylinder is $\frac{3}{4} \times 50 = 37.5$ L.
   Of the five given choices, the one that is slightly less than 37.5 L is 36 L, or (D).

   ANSWER: (D)

5. Since $PQRS$ and $WXYZ$ are rectangles, then $SR = PQ = 30$ and $WX = ZY = 15$.
   Since $SX = 10$, then $WS = WX - SX = 15 - 10 = 5$.
   Thus, $WR = WS + SR = 5 + 30 = 35$.

   ANSWER: (E)

6. Since $x = 11$, $y = 8$ and $2x + 3z = 5y$, then $2 \times 11 + 3z = 5 \times 8$ or $3z = 40 - 22$.
   Therefore, $3z = 18$ and so $z = 6$.

   ANSWER: (A)

7. *Solution 1*
   Since $(x+a)(x+8) = x^2 + bx + 24$ for all x, then $x^2 + ax + 8x + 8a = x^2 + bx + 24$ or $x^2 + (a+8)x + 8a = x^2 + bx + 24$ for all x.
   Since the equation is true for all x, then the coefficients on the left side must match the coefficients on the right side.
   Therefore, $a + 8 = b$ and $8a = 24$.
   The second equation gives $a = 3$, from which the first equation gives $b = 3 + 8 = 11$.
   Finally, $a + b = 3 + 11 = 14$.

   Solution 2
   Since $(x+a)(x+8) = x^2 + bx + 24$ for all x, then the equation is true for $x = 0$ and $x = 1$.
   When $x = 0$, we obtain $(0+a)(0+8) = 0 + 0 + 24$ or $8a = 24$, which gives $a = 3$.
   When $x = 1$, we obtain $(1+3)(1+8) = 1 + b + 24$ or $36 = b + 25$, which gives $b = 11$.
   Finally, $a + b = 3 + 11 = 14$.

   ANSWER: (D)

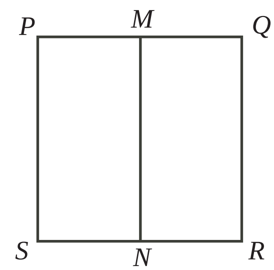
8. The original set contains 11 elements whose sum is 66.
When one number is removed, there will be 10 elements in the set.
For the average of these elements to be 6.1, their sum must be $10 \times 6.1 = 61$.
Since the sum of the original 11 elements is 66 and the sum of the remaining 10 elements is 61, then the element that has been removed is $66 - 61 = 5$.

Answer: (B)

9. Since the regular price for the bicycle is \$320 and the savings are 20%, then the amount of money that Sandra saves on the bicycle is $\$320 \times 20\% = \$320 \times 0.2 = \$64$.
Since the regular price for the helmet is \$80 and the savings are 10%, then the amount of money that Sandra saves on the helmet is $\$80 \times 10\% = \$80 \times 0.1 = \$8$.
The total of the original prices for the bicycle and helmet is $\$320 + \$80 = \$400$.
Sandra's total savings are $\$64 + \$8 = \$72$.
Therefore, her total percentage savings is $\dfrac{\$72}{\$400} \times 100\% = \dfrac{72}{4} \times 1\% = 18\%$.

Answer: (A)

10. Suppose that the side length of square $PQRS$ is x.
Then $PQ = QR = RS = SP = x$.
Since M is the midpoint of PQ, then $PM = \frac{1}{2}x$.
In terms of x, the perimeter of rectangle $PMNS$ is



$$2(PM + PS) = 2(\tfrac{1}{2}x + x) = 3x$$

(Note that $SN = PM = \frac{1}{2}x$ since N is the midpoint of RS. Also, $MN = PS = x$, since MN is parallel to PS and joins two parallel line segments.)
Since we are told that the perimeter of $PMNS$ is 36, then $3x = 36$ or $x = 12$.
Therefore, the area of square $PQRS$ is $x^2 = 144$.

Answer: (D)

11. On Monday, Ramya read $\frac{1}{5}$ of the 300 pages, which is $\frac{1}{5} \times 300 = 60$ pages in total.
After Monday, there were $300 - 60 = 240$ pages remaining to be read in the novel.
On Tuesday, Ramya read $\frac{4}{15}$ of these remaining 240 pages, or $\frac{4}{15} \times 240 = \frac{960}{15} = 64$ pages.
Therefore, she read $60 + 64 = 124$ pages in total over these two days.

Answer: (A)

12. There are 10 numbers in the list.
We note that

$$(-1)^4 = 1^4 = 1 \qquad (-3)^4 = 3^4 = 81 \qquad (-5)^4 = 5^4 = 625$$
$$(-7)^4 = 7^4 = 2401 \qquad (-9)^4 = 9^4 = 6561$$

Thus, if $m = -3, -1, 1, 3$, then $m^4 < 100$. If $m = -9, -7, -5, 5, 7, 9$, then $m^4 > 100$.
In other words, there are exactly six numbers in the list whose fourth power is larger than 100.
Thus, if m is chosen at random from the list, the probability that $m^4 > 100$ is $\frac{6}{10}$ or $\frac{3}{5}$.

Answer: (E)

13. We note that $64 = 2^6$ and $512 = 2^9$.
Therefore, the equation $512^x = 64^{240}$ can be rewritten as $(2^9)^x = (2^6)^{240}$ or $2^{9x} = 2^{6(240)}$.
Since the bases in this last equation are equal, then the exponents are equal, so $9x = 6(240)$ or $x = \frac{1440}{9} = 160$.

Answer: (D)

14. Since 25% of the money donated came from parents, then the remaining $100\% - 25\% = 75\%$ came from the teachers and students.
Since the ratio of the amount donated by teachers to the amount donated by students is $2 : 3$, then the students donated $\frac{3}{2+3} = \frac{3}{5}$ of this remaining 75%.
This means that the students donated $\frac{3}{5} \times 75\% = 45\%$ of the total amount.
Therefore, the ratio of the amount donated by parents to the amount donated by students is $25\% : 45\% = 25 : 45 = 5 : 9$.

Answer: (C)

15. Let n be the number of cookies in the cookie jar.
Let r be the number of raisins in each of the $n-1$ smaller, identical cookies.
This means that there are $r+1$ raisins in the larger cookie.
If we removed one raisin from the larger cookie, it too would have r raisins and so each of the n cookies would have the same number of raisins (r), and the total number of raisins in the cookies would be $100 - 1 = 99$.
From this, we obtain $nr = 99$.
(We could also obtain this equation by noting that there are $n-1$ cookies containing r raisins and 1 cookie containing $r+1$ raisins and 100 raisins in total, so $(n-1)r + (r+1) = 100$ or $nr - r + r + 1 = 100$ or $nr = 99$.)
Since n and r are positive integers whose product is 99, then the possibilities are:

$$99 = 99 \times 1 = 33 \times 3 = 11 \times 9 = 9 \times 11 = 3 \times 33 = 1 \times 99$$

Since n is between 5 and 10, then we must have $99 = 9 \times 11$; that is, $n = 9$ and $r = 11$.
Since there are 11 raisins in each of the smaller cookies, then there are $11 + 1 = 12$ raisins in the larger cookie.

Answer: (E)

16. Let s be the side length of each of the 60 identical squares.
Since the diagonal of each of the squares has length 2, then by the Pythagorean Theorem, $s^2 + s^2 = 2^2$ or $2s^2 = 4$, which gives $s^2 = 2$ or $s = \sqrt{2}$, since $s > 0$.
Now $PQ = 5s$ and $PS = 12s$, so since $QS > 0$, then by the Pythagorean Theorem,

$$QS = \sqrt{PQ^2 + PS^2} = \sqrt{(5s)^2 + (12s)^2} = \sqrt{25s^2 + 144s^2} = \sqrt{169s^2} = 13s$$

Since $QS = 13s$ and $s = \sqrt{2}$, then $QS = 13\sqrt{2} \approx 18.38$.
Of the given choices, this is closest to 18.

Answer: (A)

17. *Solution 1*
Suppose that the five consecutive integers represented by p, q, r, s, t are $n, n+1, n+2, n+3, n+4$, for some integer n.
The sum of any two of these integers is at most $(n+3) + (n+4) = 2n + 7$; the sum of every other pair is smaller.
The sum of any two of these integers is at least $n + (n+1) = 2n + 1$; the sum of every other pair is larger.
Therefore, the maximum possible difference between the sums of two pairs is $(2n+7) - (2n+1)$ or 6; any other choice of pairs will give a smaller difference between the sums.
Since we are told that $p + q = 63$ and $s + t = 57$, which gives $(p+q) - (s+t) = 6$, then it must be the case that p and q are the two largest integers from the list while s and t are the

two smallest integers from the list.
In other words, $p+q=(n+3)+(n+4)=63$ and so $2n+7=63$ or $2n=56$ and so $n=28$.
Since r must be the middle integer in the list, then $r=n+2=30$.

Solution 2
Suppose that the five consecutive integers represented by p,q,r,s,t are $n, n+1, n+2, n+3, n+4$, for some integer n.
The sum of all five integers is $n+(n+1)+(n+2)+(n+3)+(n+4)=5n+10$.
We are told that $p+q=63$ and $s+t=57$.
Thus, the sum of the five integers is also $p+q+r+s+t=63+r+57=120+r$.
Comparing the two expressions for the sum of the integers, we obtain $5n+10=120+r$ or $r=5n-110$.
Since $r=5n-110=5(n-22)$, then r is divisible by 5.
Of the five given answer choices, this means that we could have $r=20$ or $r=30$.
If $r=20$, then $20=5(n-22)$ or $n-22=4$ and so $n=26$. In this case, r is not one of the integers between n and $n+4$, inclusive, so r cannot be 20.
If $r=30$, then $30=5(n-22)$ or $n-22=6$ and so $n=28$. Here, the integers in the list would be $28, 29, 30, 31, 32$, which can produce the given conditions if p and q are 31 and 32, and t and s are 28 and 29.
Therefore, $r=30$.

Answer: (E)

18. Since p is a positive integer, then $p \geq 1$ and so $0 < \frac{1}{p} \leq 1$.

Since n is a positive integer, then $n \geq 1$ and so $n+\frac{1}{p} > 1$, which tells us that $0 < \dfrac{1}{n+\frac{1}{p}} < 1$.

Therefore, $m < m + \dfrac{1}{n+\frac{1}{p}} < m+1$. Since $m + \dfrac{1}{n+\frac{1}{p}} = \dfrac{17}{3}$, which is between 5 and 6, and since m is an integer, then $m=5$.

Since $m=5$, then $m + \dfrac{1}{n+\frac{1}{p}} = \dfrac{17}{3}$ gives $\dfrac{1}{n+\frac{1}{p}} = \dfrac{2}{3}$ or $n+\dfrac{1}{p} = \dfrac{3}{2}$.

Since $n < n + \frac{1}{p} \leq n+1$ and n is an integer, then $n=1$.

Thus, $n+\frac{1}{p} = \frac{3}{2}$ gives $\frac{1}{p} = \frac{1}{2}$, which gives $p=2$.
Therefore, $n=1$.

Answer: (C)

19. We rewrite the integers from the list in terms of their prime factorizations:

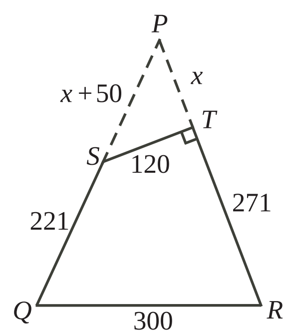
$$1, 2^1, 3^1, 2^2, 5^1, 2^1 3^1, 7^1, 2^3, 3^2$$

A positive integer larger than one is a perfect square if and only if each of its prime factors occurs an even number of times.
Since the integers in the list above contain in total only one factor of 5 and one factor of 7, then neither 5 nor 7 can be chosen to form a product that is a perfect square.
This leaves us with seven integers $1, 2^1, 3^1, 2^2, 2^1 3^1, 2^3, 3^2$, from which we need to choose six.

When the seven given integers are multiplied together, their product is $2^{1+2+1+3}3^{1+1+2} = 2^7 3^4$.
We can think of choosing six of the seven numbers and multiplying them together as choosing all seven and then dividing out the one we did not choose.
To divide the product $2^7 3^4$ by one of the integers to obtain a perfect square, the divisor must include an odd number of factors of 2 (since the product of all seven includes an odd number of factors of 2) and an even number of factors of 3 (since the product includes an even number of factors of 3). (Note that "an even number of factors of 3" includes the possibility of zero factors of 3.)
There are two such numbers in the list: 2^1 and 2^3.
(Alternatively, we could have divided the product by each of the seven numbers to determine which results in a perfect square.)
Therefore, the two sets of six numbers that satisfy the given conditions should be $1, 3^1, 2^2, 2^1 3^1, 2^3, 3^2$ (whose product is $2^6 3^4$) and $1, 2^1, 3^1, 2^2, 2^1 3^1, 3^2$ (whose product is $2^4 3^4$).
Thus, we can set $m^2 = 2^6 3^4$, which gives $m = 2^3 3^2 = 72$, and $n^2 = 2^4 3^4$, which gives $n = 2^2 3^2 = 36$.
Finally, $m + n = 72 + 36 = 108$.

Answer: (A)

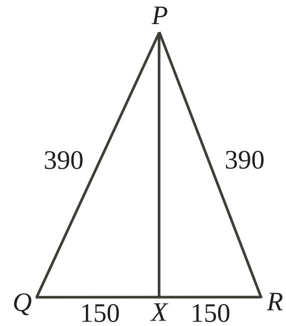
20. We calculate the area of quadrilateral $STRQ$ by subtracting the area of $\triangle PTS$ from the area of $\triangle PQR$.
Let $PT = x$.
Then $PR = PT + TR = x + 271$.
Since $PQ = PR = x + 271$ and $SQ = 221$, then $PS = PQ - SQ = (x + 271) - 221 = x + 50$.



By the Pythagorean Theorem in $\triangle PTS$, we have

$$\begin{aligned}
PT^2 + TS^2 &= PS^2 \\
x^2 + 120^2 &= (x + 50)^2 \\
x^2 + 14400 &= x^2 + 100x + 2500 \\
11900 &= 100x \\
x &= 119
\end{aligned}$$

Therefore, $\triangle PTS$ has $PT = x = 119$, $TS = 120$, and $PS = x + 50 = 169$.
Since $\triangle PTS$ is right-angled at T, then its area is $\frac{1}{2}(PT)(TS) = \frac{1}{2}(119)(120) = 7140$.
Furthermore, in $\triangle PQR$, we have $PR = PQ = x + 271 = 390$.
Now, $\triangle PQR$ is isosceles, so when we draw a median PX from P to the midpoint X of QR, it is perpendicular to QR.



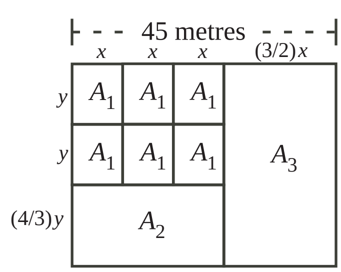
Since X is the midpoint of QR and $QR = 300$, then $QX = \frac{1}{2}QR = 150$.
We can use the Pythagorean Theorem in $\triangle PXQ$ to conclude that

$$PX = \sqrt{PQ^2 - QX^2} = \sqrt{390^2 - 150^2} = \sqrt{21600} = 360$$

since $PX > 0$.
Since PX is a height in $\triangle PQR$, then the area of $\triangle PQR$ is $\frac{1}{2}(QR)(PX) = \frac{1}{2}(300)(360) = 54\,000$.
Finally, the area of $STRQ$ is the difference in the areas of these two triangles, or $54\,000 - 7140$, which equals $46\,860$.

Answer: (C)

21. We refer to distances in the horizontal direction as widths and distances in the vertical direction as lengths.
Suppose that each of the six enclosures labelled A_1 have width x m and length y m.
Then each of these has area xy m^2.
We start by determining the dimensions of the remaining enclosures in terms of these two variables.
Enclosure A_2 has width $x + x + x = 3x$ m.
Since the area of enclosure A_2 is four times that of A_1, then its area is $4xy$ m^2.
Therefore, the length of the enclosure A_2 is its area divided by its width, or $\dfrac{4xy}{3x} = \dfrac{4}{3}y$ m. (We use the notation $(4/3)y$ in the diagram.)
Thus, the length of enclosure A_3 is $y + y + \frac{4}{3}y = \frac{10}{3}y$ m.
Since the area of enclosure A_3 is $5xy$ m^2, then its width is $\dfrac{5xy}{\frac{10}{3}y} = \dfrac{3}{2}x$ m. (We use the notation $(3/2)x$ in the diagram.)



The total width of the field is 45 m. This can also be expressed (using the top fence) as $x + x + x + \frac{3}{2}x = \frac{9}{2}x$ m.
Since $\frac{9}{2}x = 45$, then $x = \frac{2}{9}(45) = 10$.
In terms of x, the total length, in metres, of "horizontal" fencing is

$$(x + x + x + \tfrac{3}{2}x) + (x + x + x) + (x + x + x) + (x + x + x + \tfrac{3}{2}x) = 15x$$

which we calculate by going from left to right along each row from top to bottom.
In terms of y, the total length, in metres, of "vertical" fencing is

$$(y + y + \tfrac{4}{3}y) + (y + y) + (y + y) + (y + y + \tfrac{4}{3}y) + (y + y + \tfrac{4}{3}y) = 14y$$

which we calculate by going from top to bottom along each column from left to right.
Since the total length of fencing is 360 m, then $15x + 14y = 360$.
Since $x = 10$, then $150 + 14y = 360$ or $14y = 210$ and so $y = 15$.
Therefore, the area of enclosure A_1 is $xy = (10)(15) = 150$ m^2.
Of the given answers, this is closest to (in fact, equal to) 150.0.

Answer: (B)

22. Suppose that Megan and Shana competed in exactly n races.
Since Shana won exactly 2 races, then Megan won exactly $n - 2$ races.
Since Shana won 2 races and lost $n - 2$ races, then she received $2x + (n - 2)y$ coins.
Thus, $2x + (n - 2)y = 35$.
Since Megan won $n - 2$ races and lost 2 races, then she received $(n - 2)x + 2y$ coins.
Thus, $(n - 2)x + 2y = 42$.
If we add these two equations, we obtain $(2x + (n - 2)y) + ((n - 2)x + 2y) = 35 + 42$ or $nx + ny = 77$ or $n(x + y) = 77$.
Since n, x and y are positive integers, then n is a positive divisor of 77, so $n = 1, 7, 11$ or 77.
Subtracting $2x + (n - 2)y = 35$ from $(n - 2)x + 2y = 42$, we obtain

$$((n - 2)x + 2y) - (2x + (n - 2)y) = 42 - 35$$

or $(n - 4)x + (4 - n)y = 7$ or $(n - 4)(x - y) = 7$.
Since n, x and y are positive integers and $x > y$, then $n-4$ is a positive divisor of 7, so $n-4 = 1$ or $n - 4 = 7$, giving $n = 5$ or $n = 11$.
Comparing the two lists, we determine that n must be 11.
Thus, we have $11(x + y) = 77$ or $x + y = 7$.
Also, $7(x - y) = 7$ so $x - y = 1$.
Adding these last two equations, we obtain $(x + y) + (x - y) = 7 + 1$ or $2x = 8$, and so $x = 4$.
(Checking, if $x = 4$, then $y = 3$. Since $n = 11$, then Megan won 9 races and Shana won 2 races. Megan should receive $9(4) + 2(3) = 42$ coins and Shana should receive $2(4) + 9(3) = 35$ coins, which agrees with the given information.)

Answer: (E)

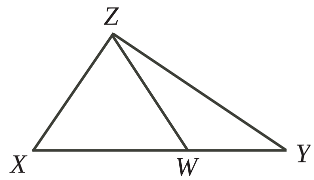
23. First, we consider the first bag, which contains a total of $2 + 2 = 4$ marbles.
There are 4 possible marbles that can be drawn first, leaving 3 possible marbles that can be drawn second. This gives a total of $4 \times 3 = 12$ ways of drawing two marbles.
For both marbles to be red, there are 2 possible marbles (either red marble) that can be drawn first, and 1 marble that must be drawn second (the remaining red marble). This gives a total of $2 \times 1 = 2$ ways of drawing two red marbles.
For both marbles to be blue, there are 2 possible marbles that can be drawn first, and 1 marble that must be drawn second. This gives a total of $2 \times 1 = 2$ ways of drawing two blue marbles.
Therefore, the probability of drawing two marbles of the same colour from the first bag is the total number of ways of drawing two marbles of the same colour $(2 + 2 = 4)$ divided by the total number of ways of drawing two marbles (12), or $\dfrac{4}{12} = \dfrac{1}{3}$.
Second, we consider the second bag, which contains a total of $2 + 2 + g = g + 4$ marbles.
There are $g + 4$ possible marbles that can be drawn first, leaving $g + 3$ possible marbles that can be drawn second. This gives a total of $(g + 4)(g + 3)$ ways of drawing two marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two red marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two blue marbles.
For both marbles to be green, there are g possible marbles that can be drawn first, and $g - 1$ marbles that must be drawn second. This gives a total of $g(g - 1)$ ways of drawing two green marbles.
Therefore, the probability of drawing two marbles of the same colour from the second bag is the total number of ways of drawing two marbles of the same colour $(2+2+g(g-1) = g^2 - g + 4)$ divided by the total number of ways of drawing two marbles $((g+4)(g+3))$, or $\dfrac{g^2 - g + 4}{(g + 4)(g + 3)}$.
Since the two probabilities that we have calculated are to be equal and $g \neq 0$, then

$$\begin{aligned}
\frac{1}{3} &= \frac{g^2 - g + 4}{(g + 4)(g + 3)} \\
(g + 4)(g + 3) &= 3g^2 - 3g + 12 \\
g^2 + 7g + 12 &= 3g^2 - 3g + 12 \\
10g &= 2g^2 \\
0 &= 2g^2 - 10g \\
0 &= 2g(g - 5)
\end{aligned}$$

Therefore, $g = 0$ or $g = 5$. Since $g \neq 0$, then $g = 5$.

Answer: (B)

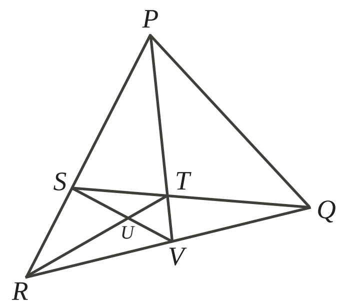
24. In this solution, we use the notation $|\triangle XYZ|$ to denote the area of $\triangle XYZ$.
In this solution, we also use a fact about a triangle (which we call $\triangle XYZ$) that is divided into two pieces by a line segment (ZW):



$$\frac{|\triangle ZXW|}{|\triangle ZWY|} = \frac{XW}{WY}$$

We label this fact $(*)$. $(*)$ is true because these triangles have a common height (the perpendicular distance, h, from Z to XY), and so $\dfrac{|\triangle ZXW|}{|\triangle ZWY|} = \dfrac{\frac{1}{2}(XW)h}{\frac{1}{2}(WY)h} = \dfrac{XW}{WY}$.

We redraw the given diagram, removing line segments PU and QU:

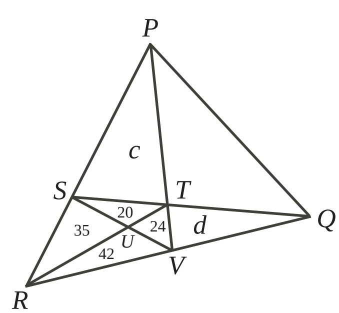


Suppose that $|\triangle SUT| = a$.
Since $|\triangle RST| = 55$, then $|\triangle RSU| = |\triangle RST| - |\triangle SUT| = 55 - a$.
Since $|\triangle RSV| = 77$, then $|\triangle RUV| = |\triangle RSV| - |\triangle RSU| = 77 - (55 - a) = 22 + a$.
Since $|\triangle RTV| = 66$, then $|\triangle TUV| = |\triangle RTV| - |\triangle RUV| = 66 - (22 + a) = 44 - a$.
By $(*)$, $\dfrac{|\triangle SUT|}{|\triangle RSU|} = \dfrac{TU}{UR} = \dfrac{|\triangle TUV|}{|\triangle RUV|}$.
Therefore,

$$\begin{aligned} \frac{a}{55-a} &= \frac{44-a}{22+a} \\ a(22+a) &= (44-a)(55-a) \\ a^2 + 22a &= 2420 - 99a + a^2 \\ 121a &= 2420 \\ a &= 20 \end{aligned}$$

Thus, $|\triangle SUT| = 20$, $|\triangle RSU| = 35$, $|\triangle RUV| = 42$, and $|\triangle TUV| = 24$.

Let $|\triangle PST| = c$ and $|\triangle QTV| = d$. We have the following configuration:



By $(*)$, $\dfrac{|\triangle PST|}{|\triangle RST|} = \dfrac{PS}{SR} = \dfrac{|\triangle PSV|}{|\triangle RSV|}$.

Therefore, $\dfrac{c}{55} = \dfrac{c+44}{77}$ or $77c = 55c + 55(44)$, which gives $22c = 2420$ or $c = 110$.

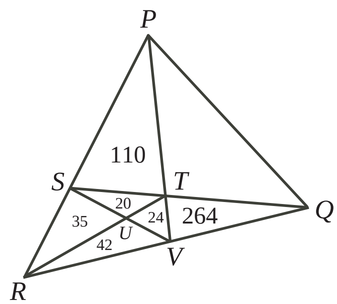
Thus, $\dfrac{PS}{SR} = \dfrac{|\triangle PST|}{|\triangle RST|} = \dfrac{110}{55} = 2$. (We'll use this later.)

Similarly, by $(*)$, $\dfrac{|\triangle QTV|}{|\triangle RTV|} = \dfrac{QV}{VR} = \dfrac{|\triangle SVQ|}{|\triangle RSV|}$.

Therefore, $\dfrac{d}{66} = \dfrac{d+44}{77}$ or $77d = 66d + 66(44)$, which gives $11d = 2904$ or $d = 264$.

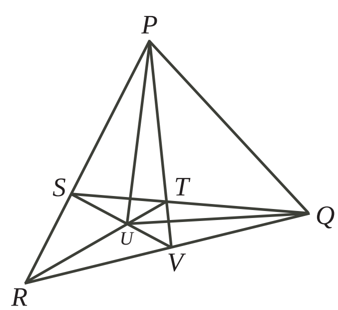
Thus, $\dfrac{QV}{VR} = \dfrac{|\triangle QTV|}{|\triangle RTV|} = \dfrac{264}{66} = 4$. (We'll use this later.)

This gives:



By $(*)$, $\dfrac{|\triangle PTQ|}{|\triangle PST|} = \dfrac{TQ}{ST} = \dfrac{|\triangle QTV|}{|\triangle STV|}$, and so $\dfrac{|\triangle PTQ|}{110} = \dfrac{264}{44}$ or $|\triangle PTQ| = \dfrac{110(264)}{44} = 660$.

We are now ready to calculate the area of $\triangle PQU$, so we add back in segments PU and QU.



We calculate this area using

$$|\triangle PQU| = |\triangle PTQ| + |\triangle PST| + |\triangle QTV| + |\triangle STV| - |\triangle PSU| - |\triangle QVU|$$

Using $(*)$ and the facts that $\dfrac{PS}{SR} = 2$ and $\dfrac{QV}{VR} = 4$, we have

$$|\triangle PSU| = \frac{PS}{SR}|\triangle RSU| = 2(35) = 70$$

and

$$|\triangle QVU| = \frac{QV}{VR}|\triangle VRU| = 4(42) = 168$$

Therefore, $|\triangle PQU| = 660 + 110 + 264 + 44 - 70 - 168 = 840$.

Answer: (C)

25. Step 1: Using parity and properties of powers of 2 to simplify the equation

We note that if $2^x = 2^y$ for some real numbers x and y, then $x = y$.
This is because $2^x = 2^y$ implies $\frac{2^x}{2^y} = 1$ or $2^{x-y} = 1$, and so $x - y = 0$ or $x = y$.
We examine equations of the form $2^a + 2^b = 2^c + 2^d$ where a, b, c, and d are integers.
(This is more general than the given equation, but allows us to determine what is possible.)
We may assume without loss of generality that $a \leq b$ and $c \leq d$ and $a \leq c$. (We can always switch the variable names to make these true.)
We factor the equation as $2^a(1 + 2^{b-a}) = 2^c(1 + 2^{d-c})$, and then divide both sides by 2^a to obtain $1 + 2^{b-a} = 2^{c-a}(1 + 2^{d-c})$.
We show that $c = a$ by contradiction:

> If $c \neq a$, then $c \geq a$ gives $c > a$.
> If $c > a$, then $c - a > 0$, so $c - a \geq 1$, since $c - a$ is an integer.
> Therefore, the right side has a factor of 2^{c-a}, so the right side is even.
> Thus, the left side is even too, which means that 2^{b-a} must be an odd integer.
> For 2^{b-a} to be an odd integer, we must have $2^{b-a} = 1$ and so $b - a = 0$ or $b = a$.
> In this case, the left side equals 2 and the right side is greater than 2, since $2^{c-a} \geq 2$ and $1 + 2^{d-c} > 1$. This is a contradiction.
> Therefore, $c = a$.

Since $a = c$, then $2^a + 2^b = 2^c + 2^d$ becomes $2^b = 2^d$ and so $b = d$.
Therefore, if $2^a + 2^b = 2^c + 2^d$ with a, b, c, d integers, then either $a = b = c = d$ or $a = c$ and $b = d$ (with $a \neq b$) or $a = d$ and $b = c$ (with $a \neq b$).

We examine these three possibilities in the given equation, noting that m, n and k are all positive integers:

- *Case 1:* $4m^2 = m^2 - n^2 + 4 = k + 4 = 3m^2 + n^2 + k$
  From the last equality, we obtain $3m^2 + n^2 = 4$.
  Since m, n are positive integers, then $m^2 \geq 1$ and $n^2 \geq 1$.
  Since $3m^2 + n^2 = 4$, then it must be that $m = n = 1$.
  Thus, $4m^2 = k + 4$ implies $4 = k + 4$ or $k = 0$.
  But $k > 0$, so this case is not possible.
- *Case 2:* $4m^2 = k + 4$ and $m^2 - n^2 + 4 = 3m^2 + n^2 + k$ and $4m^2 \neq m^2 - n^2 + 4$
  From the second equality, we obtain $2m^2 + 2n^2 + k = 4$, which is not possible since $m, n, k > 0$, and so $2m^2 + 2n^2 + k \geq 5$.
  Therefore, this case is not possible.
- *Case 3:* $4m^2 = 3m^2 + n^2 + k$ and $m^2 - n^2 + 4 = k + 4$ and $4m^2 \neq m^2 - n^2 + 4$
  The first equality rearranges to $m^2 - n^2 = k$.
  The second equality also rearranges to $m^2 - n^2 = k$.
  The last statement is equivalent to $3m^2 + n^2 \neq 4$. As we saw in Case 1, this means that (m, n) cannot be the pair $(1, 1)$, which is consistent with $m^2 - n^2 = k$ and $k > 0$.

Therefore, having examined all of the cases, we have reduced the original problem to finding the number of odd integers k between 0 and 100 for which the equation $m^2 - n^2 = k$ has exactly two pairs of positive integers (m, n) that are solutions.

Step 2: Connecting solutions to $m^2 - n^2 = k$ with factorizations of k

We can factor the left side of this equation to give $(m + n)(m - n) = k$.
Since m, n and k are positive integers, then $m + n > 0$ and $k > 0$ so $m - n > 0$, or $m > n$.
Since k is odd and each of $m+n$ and $m-n$ is an integer, then each of $m+n$ and $m-n$ is odd (since if either was even, then their product would be even).
Also, we note that $m + n > m - n$ since $n > 0$.
Suppose that (m, n) is a solution of the equation $m^2 - n^2 = k$ with $m + n = a$ and $m - n = b$ for some odd positive integers a and b with $a > b$.
Then $ab = k$, so ab is a factorization of k.
Therefore, the solution (m, n) corresponds to a specific factorization of k.
Now suppose that we start with a factorization $k = AB$ where A and B are odd positive integers with $A \geq B$.
If we try setting $m+n = A$ and $m-n = B$, then we can add these equations to give $2m = A+B$ (or $m = \frac{1}{2}(A + B)$) and subtract them to give $2n = A - B$ (or $n = \frac{1}{2}(A - B)$). Note that since $n > 0$, then $A > B$.
Therefore, every factorization of k as the product of two odd positive integers A and B with $A > B$ gives a solution to the equation $m^2 - n^2 = k$.
Since each solution gives a factorization and each factorization gives a solution, then the number of solutions equals the number of factorizations.
Therefore, we have reduced the original problem to finding the number of odd integers k between 0 and 100 which have exactly two factorizations as the product of distinct odd integers a and b with $a > b$.

Step 3: Counting the values of k

Since k is odd, then all of its prime factors are odd.
Since $k < 100$, then k cannot have three or more distinct odd prime factors, because the smallest possible product of three distinct odd prime factors is $3 \cdot 5 \cdot 7 = 105$.
Thus, k has two or fewer distinct prime factors.
If $k = pq$ for distinct primes $p < q$, then the divisors of k are $1, p, q, pq$, so k has exactly two factorizations of the desired type (namely $1 \cdot pq$ and $p \cdot q$).
Since $k < 100$ and $p \geq 3$, then $q < \frac{100}{3}$. Since q is an integer, then $q \leq 33$.
The odd primes less than 33 are $3, 5, 7, 11, 13, 17, 19, 23, 29, 31$.
If $p \geq 11$, then $pq > 11^2 = 121$, which is larger than 100.
Therefore, p can only be 3, 5 or 7.
If $p = 3$, there are 9 possible values for q (primes from 5 to 31).
If $p = 5$, there are 5 possible values for q (primes from 7 to 19).
If $p = 7$, there are 2 possible values for q (11 and 13).
Thus, there are $9 + 5 + 2 = 16$ values of k of this form that work.

If $k = p^r q^s$ with r and s positive integers and at least one of r or s is larger than 1, then k will have at least three factorizations. (For example, if $r > 1$, then $k = 1 \cdot p^r q^s = p \cdot p^{r-1} q^s = p^r \cdot q^s$ and all of these are distinct.)

If $k = p$ or $k = p^2$ with p an odd prime, then k has only one factorization as the product of distinct factors ($1 \cdot p$ and $1 \cdot p^2$, respectively). Thus, k cannot be of this form.

If $k = p^3$ with p an odd prime, then the divisors of k are $1, p, p^2, p^3$, so it has exactly two factorizations of the desired type (namely $1 \cdot p^3$ and $p \cdot p^2$).
Since $k < 100$, then p can only equal 3 (because $5^3 > 100$).
Thus, there is 1 value of k of this form that works.

If $k = p^4$ with p an odd prime, then the divisors of k are $1, p, p^2, p^3, p^4$, so it has exactly two factorizations of the desired type (namely $1 \cdot p^4$ and $p \cdot p^3$). In this case, k has a third factorization, but it is of the wrong type since the two factors will be equal.
Since $k < 100$, then p can only equal 3 (because $5^4 > 100$).
Thus, there is 1 value of k of this form that works.

If k has more than 4 factors of p, then k will have at least three factorizations of the desired type, so k cannot be of this form. (In particular, if $k = p^n$ and $n > 4$, then $k = 1 \cdot p^n = p \cdot p^{n-1} = p^2 \cdot p^{n-2}$ and these are all distinct since $n - 2 > 2$.)

Having examined all of the possible forms, we see that there are $16 + 1 + 1 = 18$ values of k that work, and so there are 18 positive integer solutions to the original equation.

Answer: (D)