The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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Hypatia Contest
(Grade 11)
Wednesday, April 13, 2016
(in North America and South America)
Thursday, April 14, 2016
(outside of North America and South America)

Time: 75 minutes

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Do not open this booklet until instructed to do so.

Number of questions: 4 Each question is worth 10 marks

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:
1. SHORT ANSWER parts indicated by •
   • worth 2 or 3 marks each
   • full marks given for a correct answer which is placed in the box
   • part marks awarded only if relevant work is shown in the space provided

2. FULL SOLUTION parts indicated by
   • worth the remainder of the 10 marks for the question
   • must be written in the appropriate location in the answer booklet
   • marks awarded for completeness, clarity, and style of presentation
   • a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

• Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
• Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as 4.14... or 1.41..., except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.
1. Raisins are sold by the scoop, cup, jar, basket, or tub in the following proportions:
   5 scoops of raisins fill 1 jar, 3 scoops of raisins fill 1 cup, 5 baskets of raisins fill 2 tubs,
   and 30 jars of raisins fill 1 tub.
   (a) How many tubs of raisins fill 30 baskets?
   (b) How many cups of raisins fill 6 jars?
   (c) Determine how many cups of raisins fill 1 basket.

2. If a line segment is drawn from the centre of a circle to the midpoint of a chord, it is
   perpendicular to that chord. For example, in Figure 1, $OM$ is perpendicular to chord $AB$.
   If a line segment is drawn from the centre of a circle and is perpendicular to a chord,
   it passes through the midpoint of that chord. For example, in Figure 2, $PR = QR$.
   (a) In the diagram, a circle with radius 13 has a chord $AB$ with length 10. If $M$ is the
       midpoint of $AB$, what is the length of $OM$?
   (b) In a circle with radius 25, a chord is drawn so that its perpendicular distance
       from the centre of the circle is 7. What is the length of this chord?
   (c) In the diagram, the radius of the circle is 65. Two parallel chords $ST$ and $UV$ are drawn
       so that the perpendicular distance between the chords is 72 ($MN = 72$). If $MN$ passes
       through the centre of the circle $O$, and $ST$ has length 112, determine the length of $UV$. 


3. For a positive integer $n$, $f(n)$ is defined as the exponent of the largest power of 3 that divides $n$.

For example, $f(126) = 2$ since $126 = 3^2 \times 14$ so $3^2$ divides 126, but $3^3$ does not.

(a) What is the value of $f(405)$?

(b) What is the value of $f(1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10)$?

(c) Let $N$ be the positive integer equal to $\frac{100!}{50!20!}$. Determine the value of $f(N)$.

(Note: If $m$ is a positive integer, $m!$ represents the product of the integers from 1 to $m$, inclusive. For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.)

(d) Given that $f(a) = 8$ and $f(b) = 7$, determine all possible values of $f(a + b)$.

4. Erin’s Pizza (EP) and Lino’s Pizza (LP) are located next door to each other. Each day, each of 100 customers buys one whole pizza from one of the restaurants. The price of a pizza at each restaurant is set each day and is always a multiple of 10 cents. If the two restaurants charge the same price, half of the 100 customers will go to each restaurant. For every 10 cents that one restaurant’s price is higher than the other restaurant’s price, it loses one customer to the other restaurant. The cost for each restaurant to make a pizza is $5.00.

As an example, if EP charges $8.00 per pizza and LP charges $9.00 per pizza, the number of customers and the resulting profit for each restaurant is shown in the table below.

<table>
<thead>
<tr>
<th>Restaurant</th>
<th>Price per pizza</th>
<th>Number of customers</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>EP</td>
<td>$8.00</td>
<td>$50 + 10 = 60</td>
<td>$60 \times ($8.00 - $5.00) = $180</td>
</tr>
<tr>
<td>LP</td>
<td>$9.00</td>
<td>$50 - 10 = 40</td>
<td>$40 \times ($9.00 - $5.00) = $160</td>
</tr>
</tbody>
</table>

(a) On Monday, EP charges $7.70 for a pizza and LP charges $9.30.

   (i) How many customers does LP have?

   (ii) What is LP’s total profit?

(b) EP sets its price first and then LP sets its price. On Tuesday, EP charges $7.20 per pizza. What should LP’s price be in order to maximize its profit?

(c) On Wednesday, EP realizes what LP is doing: LP is maximizing its profit by setting its price after EP’s price is set. EP continues to set its price first and sets a price that is a multiple of 20 cents. LP’s price is still a multiple of 10 cents and the number of customers at each restaurant still follows the rule above. Determine the two prices that EP could charge in order to maximize its profit. State LP’s profit in each case.
For students...

Thank you for writing the 2016 Hypatia Contest! Each year, more than 220,000 students from more than 60 countries register to write the CEMC’s Contests.

Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2016.

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For teachers...

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