The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *Galois Contest*

## *(Grade 10)*

***Thursday, April 12, 2018***
***(in North America and South America)***

***Friday, April 13, 2018***
***(outside of North America and South America)***

UNIVERSITY OF WATERLOO

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**Time:** 75 minutes 

*Do not open this booklet until instructed to do so.*

**Number of questions: 4** **Each question is worth 10 marks**

**Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.**

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by 
   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by 
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

**WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.**

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

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*Do not discuss the problems or solutions from this contest online for the next 48 hours.*

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*The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.*

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked  , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked  , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the $x$-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

1.  (a) Given that $x \neq 0$, simplify the expression $\dfrac{12x^2}{3x}$.

 (b) What is the value of the expression $\dfrac{12x^2}{3x}$ when $x = 5$?

 (c) Given that $n = 2m$ and $m \neq 0$, what is the value of the expression $\dfrac{8mn}{3m^2}$?

 (d) If $q = 6$, determine all positive integers $p$ for which $3 \leq \dfrac{8p^2q}{5pq^2} \leq 4$.

2. Here are two facts about circles:

- If points $A$, $B$, $C$ lie on a circle so that $\angle ABC = 90°$, then $AC$ is a diameter of the circle. This means that in Figure 1, $AC$ is a diameter of the circle.
- If points $D$, $E$, $F$ lie on a circle so that $EF$ is a diameter, then $\angle EDF = 90°$. This means that in Figure 2, $\angle EDF = 90°$.

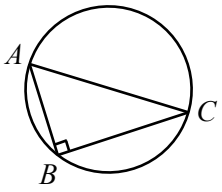

Figure 1

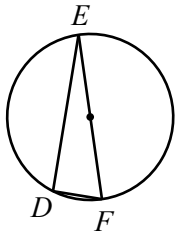

Figure 2

 (a) In Figure 1 above, $AB = 8$ and $BC = 15$. What is the length of diameter $AC$?

 (b) In Figure 2 above, $DE = 24$ and the radius of the circle is 13. What is the length of $DF$?

 (c) In Figure 3, points $P$, $Q$, $R$, and $S$ are on a circle with centre $O$. Also, $SQ$ is a diameter of the circle and $O$ is joined to $R$. If $SP = PQ$ and $\angle RQP = 80°$, determine the measure of $\angle ROQ$ and the measure of $\angle RSQ$.

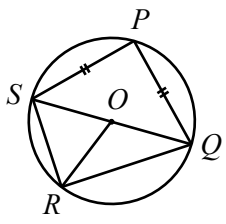

Figure 3

3. Cylinder A has radius 12 and height 25. Cylinder B has radius 9 and height $h$. Cylinder A is filled with water to a depth of 19. Cylinder B is empty. Cylinder B is lowered to the bottom of Cylinder A, as shown. Depending on the value of $h$,

(i) some water may spill out of Cylinder A onto the ground (Figure 1), or

(ii) some water may pour into Cylinder B (Figure 2), or

(iii) (i) then (ii).

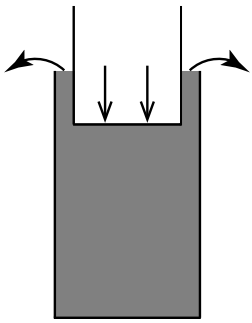

Figure 1

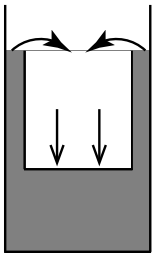

Figure 2

The walls and bases of the two cylinders are thin enough that their width can be ignored.

(a) Suppose that $h = 30$. What is the volume of water that spills out of Cylinder A onto the ground?

(b) Suppose that $h = 20$. Determine the volume of water that spills out of Cylinder A onto the ground and the depth of water in Cylinder B when it is on the bottom of Cylinder A.

(c) Determine the range of values of $h$ so that when Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full.

4. For each positive integer $k$, we define $C(k)$ to be the number of ways in which $k$ can be written as the sum of one or more consecutive positive integers. For example, $C(21) = 4$ because 21 can be written as

$$21, \quad 10 + 11, \quad 6 + 7 + 8, \quad \text{and} \quad 1 + 2 + 3 + 4 + 5 + 6,$$

and there are no other lists of one or more consecutive positive integers whose sum is 21.

(a) What is the value of $C(45)$?

(b) The positive integer $m$ equals the sum of the positive integers from 4 to $n$, inclusive. Determine the values of $a$ and $b$, with $a < b$, for which $m = \frac{1}{2}(n + a)(n + b)$ for each positive integer $n \geq 4$.

(c) Determine the value of $C(2 \times 3^4 \times 5^6)$.

(d) Determine the smallest positive integer $k$ for which $C(k) = 215$.

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Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2018.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
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**For teachers...**

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- Obtain information about our 2018/2019 contests
- Register your students for the Canadian Senior and Intermediate Mathematics Contests which will be written in November
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