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Intermediate Math Circles
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Sequences

What is a sequence?

A sequence is an ordered list of numbers.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$$

This is a finite sequence.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ..., \frac{1}{50}$$

The "..." indicates the sequence continues on up until the last term $\frac{1}{50}$. This is a finite sequence.

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$$

The "..." indicates the sequence continues without ever ending. This is an infinite sequence.

Notation:

We denote the ith term by a_i

Eg. $a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_5 = \frac{1}{5}$

The **general term** of a sequence is a formula that expresses each term in the sequence as a function of its position.

The general term for this sequence is $a_n = \frac{1}{n}$

1. **What is the next term? What is the general term?**

- $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}, \dots$

- $1, 5, 9, 13, 17, \dots$

- $3, 6, 12, 24, 48, \dots$

- $1, 8, 27, 64, 125, \dots$

- $1, -2, 4, -8, 16, \dots$

- $2, \dfrac{5}{2}, 3, \dfrac{7}{2}, 4, \dots$

- $\dfrac{3}{4}, \dfrac{5}{9}, \dfrac{7}{16}, \dfrac{9}{25}, \dfrac{11}{36}, \dots$

- $1, 2, 6, 24, 120, \dots$

- $1, 9, 29, 67, 129, \dots$

Arithmetic Sequences:

An arithmetic sequence is a sequence where the difference between any consecutive pairs of numbers in the sequence is a constant. This difference is called the **common difference**.

For example: $1, 3, 5, 7, 9, ...$ is an arithmetic sequence with first term 1 and common difference 2.

Are any of the sequences above arithmetic sequences? Which ones?

The terms in an arithmetic sequence with first term a and common difference d are:

$$a_1 = a, \quad a_2 = a + d, \quad a_3 = (a + d) + d = a + 2d, \quad a_4 = (a + 2d) + d = a + 3d, ...$$

In general, the nth term of an arithmetic sequence with first term a and common difference d is:

$$a_n = a + (n - 1)d$$

Eg., in the sequence $1, 3, 5, 7, 9, ...$, the 5th term is $a_5 = 1 + (5 - 1)2 = 9$

2. What is the 23rd term in the sequence $18, 11, 4, -3, ...$?

3. If the 4th term of an arithmetic sequence is 11 and the 8th is 31, determine the 100th.

Geometric Sequences:

A geometric sequence is a sequence where there is the same ratio between any consecutive terms. This ratio is called the **common ratio**.

For example: $1, 2, 4, 8, 16, \ldots$ is an infinite geometric sequence with first term 1 and common ratio 2.

Are any of the sequences above geometric sequences? Which ones?

The terms in a geometric sequence with first term a and common ratio r are:

$$a_1 = a, \quad a_2 = ar, \quad a_3 = (ar)r = ar^2, \quad a_4 = (ar^2)r = ar^3, \ldots$$

In general, the nth term of a geometric sequence with first term a and common ratio r is:

$$a_n = ar^{n-1}$$

Eg., in the sequence $1, 2, 4, 8, 16, \ldots$, the 5th term is $a_5 = 1 \times (2)^{5-1} = 16$

4. State the first 4 terms of the geometric sequence with general term $a_n = 2 \times (-3)^{n-2}$

5. What is the 11th term of the sequence $9, 3, 1, \frac{1}{3}, ...$?

6. The 5th term of a geometric sequence is 35 and the sixth is is 140, what is the 7th term?

Exercise Solutions:

1. next term: $\frac{1}{6}$ — general term: $a_n = \frac{1}{n}$

   next term: 21 — general term: $a_n = 1 + 4(n - 1)$

   next term: 96 — general term: $a_n = 3 \times 2^{n-1}$

   next term: 216 — general term: $a_n = n^3$

   next term: -32 — general term: $a_n = (-2)^{n-1}$

   next term: $\frac{9}{2}$ — general term: $a_n = 2 + \frac{1}{2}(n - 1)$

   next term: $\frac{13}{49}$ — general term: $a_n = \frac{2n+1}{(n+1)^2}$

   next term: 720 — general term: $a_n = n!$

   next term: 221 — general term: $a_n = n^3 + n - 1$

2. Using the formula for an arithmetic sequence we get:
   $a_{23} = 18 + (22)(-7) = -136$

3. Set up a system of equations:
   $a_4 = 11 = a + 3d$
   $a_8 = 31 = a + 7d$
   from the first equations we get that $a = 11 - 3d$. Substituting into the second equation, we get $d = 5$, and thus $a = -4$.
   Therefore the 100th term is $a_{100} = -4 + (99)5 = 491$

4. $\frac{-2}{3}, 2, -6, 18$

5. We note that the first term is 9 and the common ratio is $\frac{1}{3}$.
   Therefore we have $a_{11} = 9(\frac{1}{3})^{10} = \frac{1}{6561}$

6. From a_5 and a_6 we know that the common ratio is $\frac{140}{35} = 4$.
   We know that $a_6 = 140$, thus $a_7 = a_6 \times r = 140 \times 4 = 560$.

Extra Problems:

1. The first term of a sequence is 20.
   If a term in the sequence is t and t is even, the next term is $\frac{1}{2}t$.
   If a term in the sequence is t and t is odd, the next term is $3t + 1$.
   Therefore, the first three terms in the sequence are 20, 10, 5.
   What is the 10th term of the sequence?

2. You start your new after-school job making $8.90/hour. Your boss promises you an increase of $0.15/hour after each month of work. How long will it take until you are making at least twice your starting wage?

3. Determine the largest four-digit number to be found in the arithmetic sequence $1, 4, 7, 10, 13, 16, ...$

4. In a sequence of positive numbers, each term after the first two is the sum of *all* of the previous terms. If the first term is a, the second is 2, and the sixth is 56, then determine the value of a.

5. If the first four terms of an arithmetic sequence are $a, 2a, b,$ and $a - 6 - b$ for some numbers a and b, then what is the value of the 100th term?

6. In the sequence of five numbers x, ___, 3, ___, 18, each number after the second is obtained by multiplying the two previous terms. Determine the value of x.

7. a) Find the 11th term in the arithmetic sequence 17, 22, 27, 32, ...

    b) Explain why there is no number which occurs in each of the following arithmetic sequences:
       17, 22, 27, 32, ...
       13, 28, 43, 58, ...

    c) Find a number between 400 and 420 which occurs in both of the following arithmetic sequences:
       17, 22, 27, 32, ...
       16, 22, 28, 34, ...

8. a) If $x - 1, 2x + 2$ and $7x + 1$ are the first three terms of an arithmetic sequence, determine the value of x.

    b) For the value of x from (a), what is the middle term of the arithmetic sequence
       $x - 1, 2x + 2, 7x + 1, ..., 72$?

9. In the four term sequence $0.001, 0.02, 0.4, x$, each term after the first is found by multiplying the previous term by the same number. What is the value of x?

10. A company is storing 3 kg of radioactive material until it becomes safe to the environment. After one year, 95% of the radioactive material remains. How much radioactive material will be left after 100 years?

11. How many terms are there in the geometric sequence $1024, 256, 64, ..., \dfrac{1}{64}$?

12. If the 5th term of a geometric series is 35 and the eighth is 385, what is the 20th term?

13. The sequence $6, -9, x, y$ is such that the first three terms form an arithmetic sequence and the last three terms form a geometric sequence. Find the values of x and y.