



University of Waterloo
Faculty of Mathematics



Centre for Education in
Mathematics and Computing

Senior Math Circles

February 25, 2009

Geometry I

Archimedes

A brief look into the fabulous life of Archimedes of Syracuse (287–212 B.C.).

Eccentric genius.

Concerned himself with practical matters, but could delve into the most abstract realms.

Innovator (Archimedean screw, ship shaker, etc.).

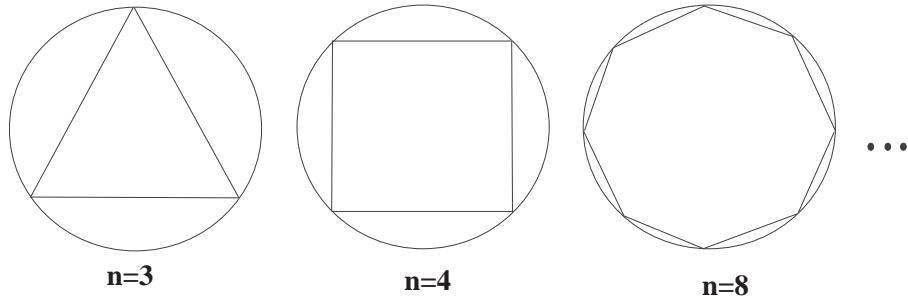
Made his most profound contributions to Mathematics (some argue the greatest mathematician of antiquity).

Mastered the method of exhaustion to approximate areas and volumes.

Area of a Circle

Archimedes used the method of exhaustion, using the area of a known figure to approximate the area of a circle.

Compute the area of a regular n -gon of radius r . This is what we use to approximate the area of a circle. This can be performed in two fashions. Using *inscribed polygons* in circle or vice versa. Both methods will converge on the same result.



As n gets larger we have a better approximation for our circle.

Exercise: Prove, using the method of exhaustion, that the area of a circle is in fact πr^2 .

Proof:

Let our circle have a radius of r .

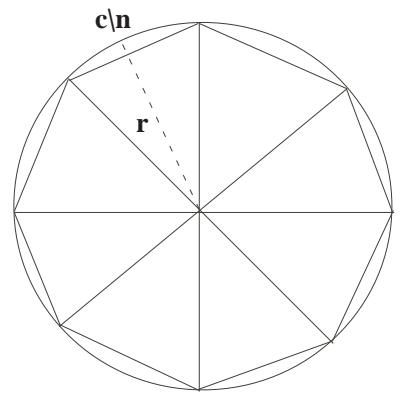
Consider the n -gon with $n = 8$. We can see that to compute the area of this figure, is the same as taking the area of n congruent triangles.

So we get that $A = 8 \times (\frac{1}{2}bh)$. Now consider when n gets really large. The area between the circle and the n -gon will decrease, giving us a better approximation.

The base of these triangles is approximately one portion of the circumference of the circle. Namely, $b = \frac{c}{n}$.

Similarly, the height of these triangles converges to the length of the radius of the circle, as n gets large.

Now we have in general that $A = n \times (\frac{1}{2} \frac{c}{n}r) = n \times \frac{1}{2} \frac{2\pi r}{n}r = \pi r^2$ as required. \square



Surface Area of a Sphere

Archimedes's Proof of the Surface Area of a Sphere:

We will take our sphere, slice out n vertical pieces of the surface area, and flatten them out. (see diagram). As we make the slices smaller, ie. n larger, the slices become more like triangles.

Again, we say that the surface area of the sphere is equal to the sum of the congruent triangles. This time we have $2n$ triangles.

$$\text{Surface Area} = 2n \times (\frac{1}{2}bh)$$

The base of each triangle is equal to the circumference of the sphere, divided by the number of slices n .

Similarly, the height of each triangle is equal to the circumference of the sphere, divided by 4.

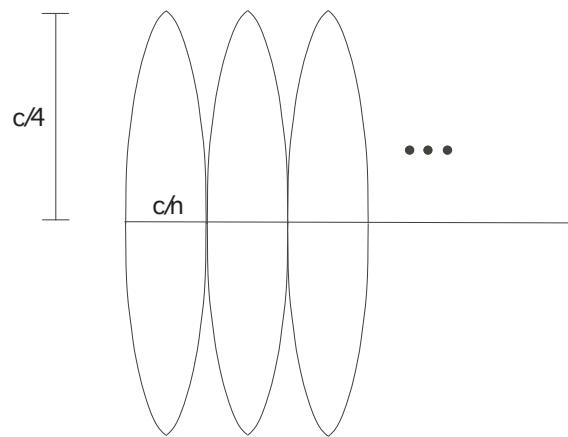
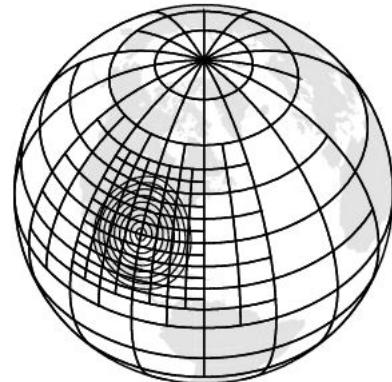
$$\text{Therefore we have } 2n \frac{1}{2} \frac{c}{n} \frac{c}{4}$$

$$= 2n \frac{1}{2} \frac{2\pi r}{n} \frac{2\pi r}{4}$$

$$= \pi^2 r^2$$

However, we know that the surface area of a sphere is actually $4\pi r^2$. Archimedes was off by a factor of $\frac{4}{\pi}$.

Where did Archimedes go wrong?



Heron's Formula

If the area of the triangle is K , and $s = \frac{a+b+c}{2}$ is the semi-perimeter, then

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Exercise: Prove Heron's Formula.

Proof:

Inscribe a circle inside $\triangle ABC$. Draw the line segments from the vertices of the triangle, through the center of the circle, intersecting with the sides of the triangle at D, E and F .

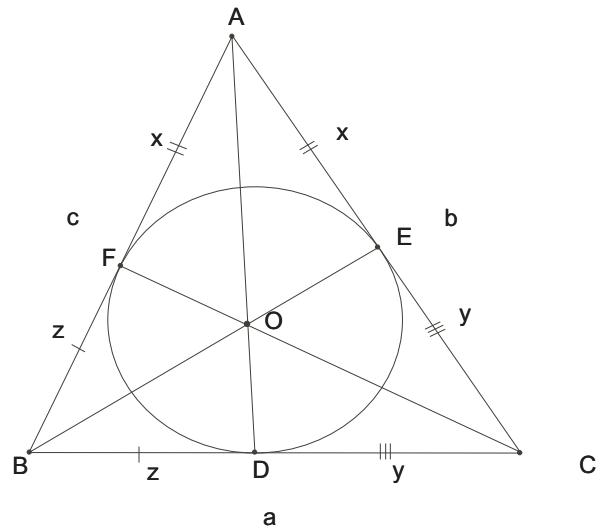
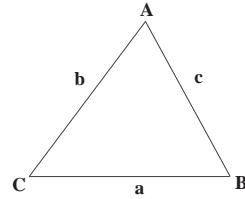
Consider $\triangle BFO$ and $\triangle BDO$. Both share the common side BO , while FO and DO are equal since they are both radii, and the third line segment extends to point B in both. Therefore the triangles are congruent. A similar argument can be made for the other triangles. Therefore we can say that $BF = BD, DC = EC$ and $AE = FA$. Call these sides x, y, z .

We can now say that $a = z + y, b = x + y$, and $c = z + x$.

We also know that $S = \frac{a+b+c}{2} = \frac{x+y+x+z+z+y}{2} = x + y + z$.

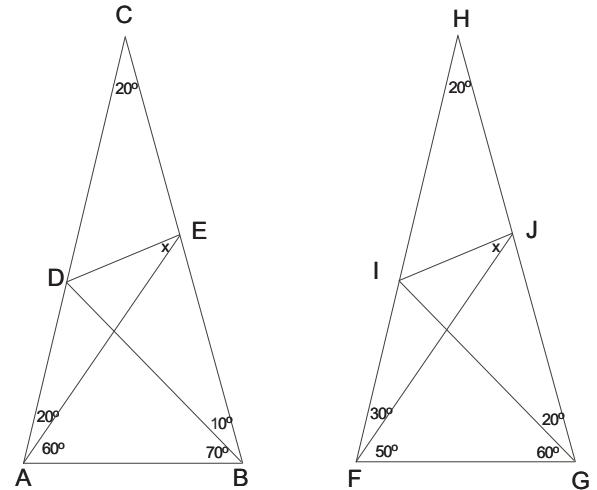
Finally, $K = 2(\text{sum of the areas of our 6 triangles}) = 2(\frac{1}{2}xr + \frac{1}{2}yr + \frac{1}{2}zr) = (x + y + z)r = sr$

Next lesson we will show that $sr = \sqrt{s(s-a)(s-b)(s-c)}$



Problems

1. Find the angle x in the following two triangles. Note that the same method will not work for both.



2. Robinhood problem:

Late at night, Robinhood takes a flashlight and climbs exactly halfway up a ladder in order to break into a house. Unfortunately (or fortunately, depending on your point of view), he hasn't secured the base of the ladder. The ladder slides out from under him, leaving him ignominiously on the ground.

A police officer running to the scene observes Robinhood's attempt. Through the darkness, all she can see is the path of the flashlight. However, she is able to deduce from this the nature of the misadventure and later provides a surprisingly accurate testimony at Robinhood's trial.

What path did the flashlight take? did it travel in a straight line between point A (Robinhood's spot halfway up the ladder) and point B (Robinhood's resting place after the fall)? What would be the path of a point on the ladder but not at the halfway point?