UNIVERSITY OF WATERLOO
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Grade 6 Math Circles
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Probability of Games
Solution to the Exercises

1. (a) $\frac{5}{6}$
   (b) $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5^4}{6^6} \approx 1.3\%$
   (c) $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216} \approx 11.6\%$
   (d) Expected gain$= 5 \times \frac{1}{6} + 1 \times \frac{5}{6} - 2 = -\frac{1}{3} < 0$. You lose about $\frac{1}{3} \approx \$0.33$ each play on average. No this game is not worth playing.

2. (a) There are 4 ways to add to a total of 5 from two spinners: $\{1,4\},\{4,1\},\{2,3\},\{3,2\}$. There is a total of $5 \times 5 = 25$ possible combinations of the 2 wheels. Therefore, the probability that the sum of the spinners is exactly 5 is $\frac{4}{25}$.
   (b) Similar to problem 1. Since $\frac{4}{25}$ of the times, I will have exactly 5 as the sum, then $\frac{21}{25}$ of the times I will have either greater than or less than 5 as the sum. Hence, my expected gain is: $5 \times \frac{4}{25} + 1 \times \frac{21}{25} - 2 = -\frac{1}{5} \approx \$-0.20$. I lose about \$0.20 on average per play. No this game is not worth playing.

3. (a) Let's first calculate the probability of winning each amount from the wheel.
   - There is a $\frac{50^\circ \times 2}{360^\circ} = \frac{5}{18}$ probability of winning back \$1.
   - There is a $\frac{20^\circ \times 2}{360^\circ} = \frac{1}{9}$ probability of winning back \$2.
   - There is a $\frac{15^\circ \times 2}{360^\circ} = \frac{1}{12}$ probability of winning back \$5.
   - There is a $\frac{5^\circ \times 2}{360^\circ} = \frac{1}{36}$ probability of winning back \$20.

   My expected gain is: $20 \times \frac{1}{36} + 5 \times \frac{1}{12} + 2 \times \frac{1}{9} + 1 \times \frac{5}{18} - 2 = \frac{53}{36} - 2 = -\frac{19}{36} \approx -\$0.53 < 0$. I lose about \$0.53 on each play. No this game is not worth playing.

   (b) i. We need to know how much of the full circle is now white space. Initially, half the circle is white space. Now that I've lost my first toss, all the \$1 and \$2 slots are becoming white spaces. Therefore, I have $180^\circ + 100^\circ + 40^\circ = 320^\circ$ out of 360°, i.e. $\frac{8}{9}$ of the wheel is white space. So my probability of hitting the white space on the second toss is $\frac{8}{9} \approx 89\%$.
   ii. To earn the \$100, my first toss must hit the white space. The probability of my first toss hitting the white space is $\frac{1}{2}$. After my first toss hits the white space, the old \$20 slot becomes \$100 slots. Remember that these slots together make up $\frac{1}{36}$ of the wheel. So my chance of getting the \$100 is $\frac{1}{36}$ on the second toss. Hence my total probability of earning the \$100 is $\frac{1}{2} \times \frac{1}{36} = \frac{1}{72}$. I.e. I'll probably get the \$100 once in every 72 tosses. (note that I would've spent $72 \times 2 = 144$ dollars to toss the token 72 times, plus a lot of time wasted on tossing my token).

iii. I need to add up the possible winnings from both the first toss and the second toss. Expected gain= $\frac{53}{36} + (\frac{1}{72} \times 100 + \frac{1}{2} \times \frac{1}{12} \times 20) - 2 = \frac{53}{36} + \frac{30}{36} + \frac{50}{36} - 2 = \frac{133}{36} - \frac{72}{36} = \frac{61}{36} \approx \$1.69 > 0$. I earn on average \$1.69 per play. Yes this is worth playing =).

4. (a) $\frac{1}{52}$

   (b) $\frac{13}{52} = \frac{1}{12}$

   (c) There are 10 diamonds left in the deck and 48 cards total left in the deck. Hence your chance of getting a diamond is $\frac{10}{48} = \frac{5}{24}$

   (d) i. The number of ways to get 18 from 2 cards are: {8,10}, {9,9} and {10,8}. There are 4 suites with value 8, 4 suits with value 10, same goes for if the first card is 10 and the second card is 8. There are 4 suites with value 9, but there are only 3 left for the 2nd card if the first card is a 9. Therefore, the total number of ways to get the above 3 combinations are: $4 \times 4 + 4 \times 3 + 4 \times 4 = 44$. There are 52 choices for the first card and 51 choices for the 2nd card. There are a total of $52 \times 51 = 2652$ ways of getting 2 cards. Hence my probability that the 2 cards I draw will have a sum of 18 is $\frac{44}{2652} \approx 1.66\%$.

   ii. There are 49 cards left in the deck. The value of the card in my hand is currently 18. My next card must have value 3 or less. There are $11 + 4 = 15$ cards of value 1 left, 4 cards of value 2 left and 4 cards of value 3 left. Therefore, my probability that if I draw one more, my total is less than or equal to 21 is $\frac{15+4+4}{49} = \frac{23}{49} \approx 47\%$.

   (e) Not getting a Queen of Spades in all 13 draws is calculated as: $\frac{51}{52} \times \frac{50}{51} \times \frac{49}{50} \times ... \frac{39}{40} = \frac{39}{52} = \frac{3}{4}$.

   (f) (Note that this question is changed to "What is the probability that you draw **13** cards and you do get a Queen of Spades?") Remember our probability chart, that the probability of not getting something + the probability of getting something equals 1. So from the last part, we have the probability of not getting Queen of Spades is $\frac{3}{4}$. Then the probability of getting the Queen of Spades is $1 - \frac{3}{4} = \frac{1}{4}$.

5. (a) There are 50 choices for the first box, 50 choices for the second box, etc. There are 25 boxes, so I have 50 multiplied 25 times. Then I have a total number of $50 \times 50 \times 50 ... \times 50 = 50^{25}$ possible bingo cards. (It's a very large number)

   (b) There are 50 choices for the first box, 49 choices for the second box, etc. Until the 25th box in which there are 26 choices for it ($50 - 5 + 1 = 26$). So I have a total of $50 \times 49 \times 48 \times ...27 \times 26$ possible bingo cards. (Smaller than part a). but also a very large number).

6. (a) $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

   (b) The probability that a person will get anything is $\frac{1}{8}$. Expected gain= $5 \times \frac{1}{8} - 2 = \frac{5}{8} - 2 = -\frac{11}{8} \approx -\$1.38$. On average, you are expected to lose \$1.38 on each game.

   (c) $\frac{9}{10} \times 1 + \frac{1}{10} \times 2 - 2 = -\frac{9}{10} = -\$0.90$. On average, you are expected to lose \$0.90 each class.

   (d) We must add the two scenarios where scenario 1, the person does get something from the big stuffed animal pit. The scenario 2, the person doesn't get anything from the big stuffed animal pit so instead gets something from the small stuffed animal pit.
   Scenario 1: I have $\frac{1}{8} \times 5$ as the possible gain on getting an animal in the big pit
   Scenario 2: I have $\frac{7}{8} \times (\frac{1}{10} \times 2 + \frac{9}{10} \times 1)$ as the possible gain on getting an animal in the

big pit.

It costs me \$2 to play. So my total expected gain is: $\frac{1}{8} \times 5 + \frac{7}{8} \times (\frac{1}{10} \times 2 + \frac{9}{10} \times 1) - 2 = \frac{5}{8} + \frac{77}{80} - 2 = \frac{127}{80} - 2 = -\frac{33}{80} \approx \$0.40$. On average, I lose about \$0.40 cents per game.

(e) I'm expected to get a big one in every 8 games.

(f) Since I'm expected to get a big one in every 8 games, I expect about 1 big one, at most 2 big ones. If I get 1 big one, then 14 other times, my claw goes to the small pit. So I expect, that out of 14 times, one of times I will get 2 small animals in one go. Therefore, I expect about 15 small animals if I get 1 big animal and I expect about 14 small animals if I get 2 big ones. I've spent \$30 and only got back about \$20 if I get 1 big animal, and at most \$24 if I get 2 big animals. Therefore, I'm always losing money. No it's not really worth playing.