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Grade 6 Math Circles
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Divisibility

Introduction

A **factor** is a whole number that divides exactly into another number without a remainder. If a is a factor of b, where $b \geqslant a$, then we say that "b is divisible by a". In mathematical terms, we write "b is divisible by a if and only if there exists some whole number, k, such that $b = a \times k$".

Examples

1. Is 24 divisible by 4?
   Yes. 24 is divisible by 4 because $24 \div 4 = 6$. (Note: 4 and 6 are factors of 24.)

2. Is 24 divisible by 5?
   No. 24 is not divisible by 5 because $24 \div 5 = 4.8$, or 4 remainder 4.

Divisibility has many applications in elementary number theory, such as prime factorization, which you will see at the end of this lesson. Because of this, it is important to be able to recognize divisibility quickly – without a calculator!

Divisibility Tricks

Because you are all very comfortable with the 12×12 multiplication table, you should also be very quick to identify divisibility of numbers up to 144 by the number 1 through 12.

But how would you identify divisibility of much larger numbers?

Luckily, quick divisibility tests (or tricks) have been developed to solve this problem. The table on the next page outlines 11 quick divisibility tests for the numbers 2 through 12.

Divisible by:	If:
2	The last digit of the number is even (0, 2, 4, 6, or 8).
3	The sum of all the digits is divisible by 3. (Note: Repeat for large numbers.)
4	The number formed by the last two digits is divisible by 4.
5	The number ends in 5 or 0.
6	The number is divisible by 2 *and* 3. That is, the number is divisible by 6 if it passes the tests for 2 *and* 3.
7	Twice the last digit subtracted from the remaining digits is divisible by 7. (Note: Repeat for large numbers)
8	The last 3 digits are divisible by 8 Note: A 3 digit number abc is divisible by 8 if: $i)$ a is even and bc is divisible by 8, or $ii)$ a is odd and $(bc - 4)$ is divisible by 8.
9	The sum of the digits is divisible by 9. (Note: Repeat for large numbers.)
10	The number ends in 0.
11	The sum of every second digit less the remaining digits is divisible by 11. (Hint: 0 is divisible by 11. Note: Repeat for large numbers.)
12	The number is divisible by 4 *and* 3. That is, the number is divisible by 12 if it passes the tests for 3 *and* 4.

Examples

1. Is 234987 divisible by 3?

   A number is divisible by 3 if "the sum of all the digits is divisible by 3". The sum of all the digits in 234987 is $2+3+4+9+8+7=33$, and 33 *is* divisible by 3 ($33 \div 3 = 11$). Thus 234987 is divisible by 3.

2. Is 398910 divisible by 7?

   A number is divisible by 7 if "twice the last digit subtracted from the remaining digits is divisible by 7".

   On the first iteration you get $39891 - 2 \times 0 = 39891$.
   On the second iteration you get $3989 - 2 \times 1 = 3987$.
   On the third iteration you get $398 - 2 \times 7 = 384$.
   On the fourth iteration you get $38 - 2 \times 4 = 30$.
   30 *is not* divisible by 7 because $30 \div 7 = 4$ remainder 2.
   Thus 398910 is not divisible by 7.

3. Is 761673 divisible by 11?

   A number is divisible by 11 if "the sum of every second digit less the remaining digits is divisible by 11".

   To differentiate the "second digits", rewrite the number a little bit: 761673.
   Now check if $6 + 6 + 3 - 7 - 1 - 7$ is divisible by 11.
   $6 + 6 + 3 - 7 - 1 - 7 = 0$, and 0 is divisible by 11.
   Thus 761673 is divisible by 11.

Prime Numbers

A **prime number** is a whole number greater than 1 that has only two factors: itself and 1.

Prime numbers are important in mathematics because they are the building blocks of all numbers. Prime numbers are also important because they are extensively used in the study of cryptography (the science of coding and decoding messages so as to keep these messages secure).

How many primes are there?

The Greek mathematician Euclid proved that there are infinitely many prime numbers over 2300 years ago. To this day, there are computers all over the world dedicated to finding more prime numbers.

Example

List all of the primes less than 100 by completing the following steps on the grid on the next page. This is called the "Sieve of Eratosthenes".

1. Cross out the number 1 since all primes are greater than 1.
2. Circle the number 2; cross out all other multiples of 2 (all even numbers).
3. Circle the number 3; cross out all other multiples of 3.
4. Circle the number 5; cross out all other multiples of 5.
5. Circle the number 7; cross out all other multiples of 7.
6. Circle all the remaining numbers. The circled numbers are all the primes less than 100!

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime Factorization

A **composite number** is a whole number that has factors other than itself and 1. Thus composite numbers are the opposite of prime numbers.

Prime factorization is the process in which a composite number is decomposed (broken up) into the product of prime numbers.

Examples

1. Find the prime factorization of 21.

   Factor trees are a great tool for finding prime factorizations.

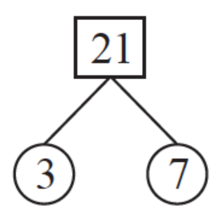
   Begin by writing the composite number of interest at the very top.

   Begin doing divisibility tricks to find factors of the composite number.

   In this case, we know that $21 = 3 \times 7$.

   Both 3 and 7 are prime numbers, so we're done!

   Thus the prime factorization of 21 is 3×7.



2. Find the prime factorization of 48.

   Begin doing divisibility tricks to find factors of 48.

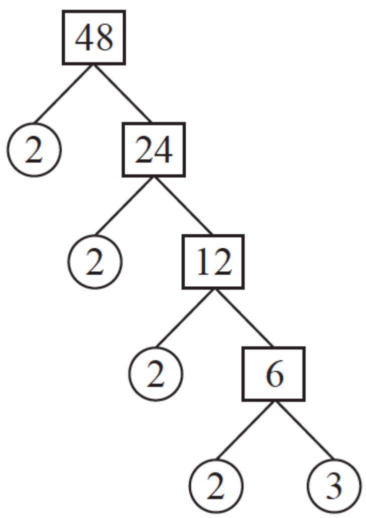
   - Because the last digit of our composite number is even, we know that 48 is divisible by 2; $48 = 2 \times 24$.

     2 is prime, but 24 is not. Thus we must find the prime factorization of 24.
   - 24 is also divisible by 2; $24 = 2 \times 12$.

     2 is prime, but 12 is not. Thus we must find the prime factorization of 12.
   - 12 is also divisible by 2; $12 = 2 \times 6$.

     2 is prime, but 6 is not. Thus we must find the prime factorization of 6.
   - 6 is also divisible by 2; $6 = 2 \times 3$.

     Both 2 and 3 are prime numbers, so we're done!



   The prime factorization is found by multiplying all the numbers at the *ends* of the tree's "branches" (the circled numbers above).

   Thus the prime factorization of 48 is $2 \times 2 \times 2 \times 2 \times 3$ or $2^4 \times 3$.

Any whole number greater than 1 is either a prime number, or can be written as a unique product of prime numbers (ignoring the order). What does this mean? In simple terms, this says that no two numbers greater than 1 have the same prime factorization. This fact is so important in mathematics that it is called *The Fundamental Theorem of Arithmetic*.

Greatest Common Factor

The **greatest common factor (GCF)** of two or more numbers is the largest factor that the numbers have in common. This can also be called the *greatest common divisor (GCD)*.

We will cover two different ways to calculate a GCF:

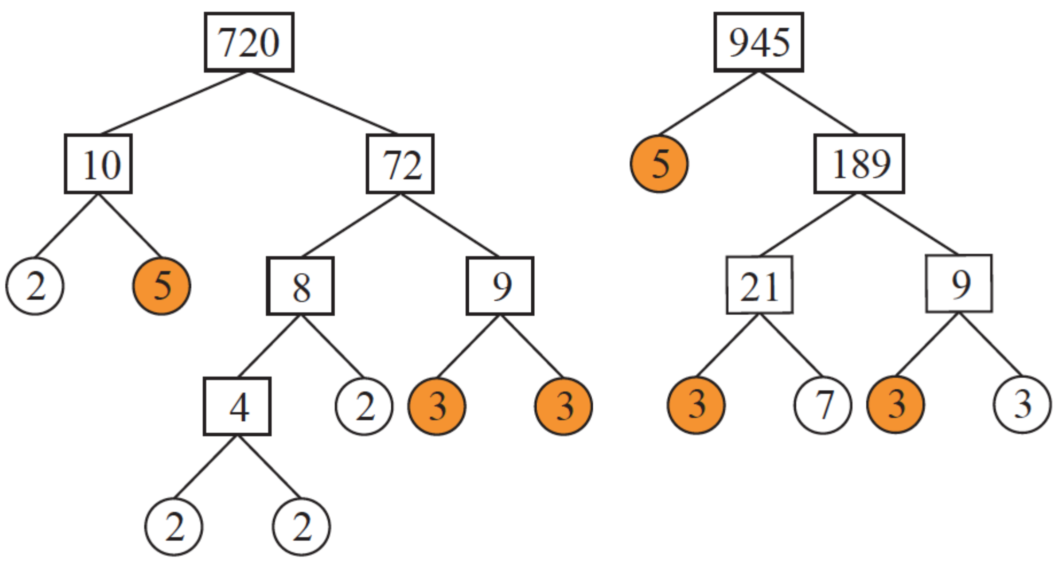
- prime factorization, and
- the Euclidean Algorithm

To find the GCF of two numbers using prime factorization, begin by constructing factor trees for both numbers. Take note of any prime factors that the numbers have in common. The GCF is the product of those prime numbers.

Example

Determine the GCF of 720 and 945.

Begin by constructing the factor trees for both numbers.



Thus $720 = 2 \times 5 \times 2 \times 2 \times 2 \times 3 \times 3$ and $945 = 5 \times 3 \times 7 \times 3 \times 3$.
To find the greatest common factor of these numbers, we look at all of the prime factors they have in common (orange above). Both numbers have two 3s and one 5 as prime factors. Thus their GCF is $3 \times 3 \times 5 = 45$.

To find the GCF of two numbers using the Euclidean Algorithm, you only need to be comfortable with long division. The algorithm is best described with an example.

Example

Determine the GCF of 720 and 945.

Begin by using long division to divide the bigger number by the smaller one.

$$\begin{array}{r} 1 \\ 720\,\overline{)\,945} \\ \underline{720} \\ 225 \end{array}$$

Now take the divisor (720) and divide it by the remainder (225).

$$\begin{array}{r} 3 \\ 225\,\overline{)\,720} \\ \underline{675} \\ 45 \end{array}$$

Again, take the divisor (225) and divide it by the remainder (45).

$$\begin{array}{r} 5 \\ 45\,\overline{)\,225} \\ \underline{225} \\ 0 \end{array}$$

Stop when the remainder is 0. The last non-zero remainder is the greatest common divisor!

The last non-zero remainder in this example is 45, thus the GCF of 720 and 945 is 45.

Problem Set

1. Using the divisibility rules, check off each of the numbers from 2 to 12 that each number is divisible by.

	2	3	4	5	6	7	8	9	10	11	12
272	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
396	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
926	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
5 940	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
6 048	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
7 848	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
22 176	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
39 208	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐
479 001 600	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐	☐

2. The number 2 has two divisors, 1 and 2. The number 4 has three divisors, 1, 2, and 4. What is the smallest number with six divisors?

3. Mr. Lawrence wants to split his Grade 6 math class of 36 students into groups for an upcoming assignment. List all the possibilities of groups, each with the same number of students, that Mr. Lawrence can divide his class into so that no students are left without a group.

4. The volume of a box is 1925 cm^3. What are the different possible dimensions of the box? (Note: The volume of a box is *length* × *width* × *height*.)

5. The eight digit number 1234☐678 is divisible by 11. What is the digit ☐?

6. Given that $29x54y214z2$ is divisible by 4 and 9, determine the values of x, y, and z.

7. How much do you need to add to 375 for it to be divisible by the following numbers?

   (a) 2 (b) 3 (c) 4 (d) 5 (e) 6 (f) 7

8. * Determine the smallest possible whole number that is divisible by 12 and consists only of 0's and 1's.

9. Create the factor trees for the following composite numbers.

   (a) 84 (b) 156 (c) 250 (d) 366

10. Draw two different factor trees for the number 180. (Start with two different pairs of factors.) What do you notice about their prime factorizations?

11. Find two numbers that have a product of 81 and a sum of 30.

12. Two numbers have a product of 300. What are all of the possible sums of these numbers?

13. * Two flashing signs are turned on at the same time. One sign flashes every 4 seconds and the other flashes every 6 seconds. How many times will the signs flash at the same time in one minute?

14. ** I am telling my friend a riddle. I tell her that the product of my three siblings ages is 36, and the sum of their ages is the day of the month on which her birthday falls. After a bit of thinking, she tells me I haven't given her enough information. I tell her that I must pick my oldest sibling up from soccer practice. How old are my three siblings?

15. ** Two numbers are *co-prime* if they do not have any prime factors in common. List all possible numbers that are co-prime with the following numbers that are < 50.

    (a) 390 (b) 210 (c) 49335

16. Determine the greatest common factors of the following numbers.

    (a) 231 and 660 (b) 1386 and 322

17. When is the GCF of two numbers a and b equal to a or b?

18. * Sally is making loot bags for a birthday party. She has 72 candy bars, 84 lollipops, and 48 gum balls. What is the largest number of identical loot bags she can make without having any candy left over? How many candy bars, lollipops, and gum balls are in each bag? (Hint: To find the GCF of three numbers, use the fact that $GCF(a, b, c) = GCF(a, GCF(b, c))$. That is, the GCF of three numbers is equal to the GCF of one of the numbers with the GCF of the other two numbers.)