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# Grade 6 Math Circles
October 8/9, 2013
## *Algebra*

**Note:** Some material and examples from the Tuesday lesson were changed for the Wednesday lesson. These notes correspond to the Wednesday lesson and are the most up-to-date/edited. No exercises or problem set questions have been changed.

# Order of Operations (BEDMAS)

## Example 1
Suppose you are asked to evaluate an expression that looks something like this

$$8 + 4 \times 3 \div 2 = ?$$

The expression above contains three operations. Can you name them?

When an expression contains many operations, we must be careful of what order we do the operations in. For example, if we do all the operations as we would read them (from left to right), we would get the following:

$$8 + 4 \times 3 \div 2 = 12 \times 3 \div 2$$
$$= 36 \div 2$$
$$= 18$$

However, in a few minutes, we will see why the correct answer is actually 14. The reason for this surprising result is BEDMAS! But what is BEDMAS?

BEDMAS is an acronym that reminds us of the correct order of operations:

**B**rackets — First Priority
**E**xponents — Second Priority
**D**ivision — Third Priority
**M**ultiplication — Third Priority
**A**ddition — Fourth Priority
**S**ubtraction — Fourth Priority

BEDMAS tells us that brackets are the highest priority, then exponents, then both division and multiplication, and finally addition and subtraction. This means that we evaluate exponents before we multiply, divide before we subtract, etc.

So going back to Example 1, we see that we made the mistake of adding before dividing. According to BEDMAS, division is a higher priority than addition.

So that was one source of error. But we also divided and multiplied in the expression. In such an expression, do we multiply or divide first? The rule to follow is this:

> If an expression has two or more operations of the same priority,
> do those operations from left to right

In Example 1, we divided first and then multiplied (left to right). So it turns out that we were lucky and followed the order of operations without knowing it! In the future, we will always follow the rule above rather than guessing.

Let's take another look at our example, and apply BEDMAS.

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## Example 1 (Revisited)

$$8 + 4 \times 3 \div 2 = 8 + 12 \div 2$$
$$= 8 + 6$$
$$= 14$$

From BEDMAS, we see that the division and multiplication must be done before the addition. We also know that we multiply before dividing in this case because the multiplication sign comes first when we read the expression from left to right.

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## Exercise I

Evaluate the following expressions:

(a) $11 - 4 + 13 \times 2 = ?$

(b) $5 - 45 \div 15 + \dfrac{3}{2} = ?$

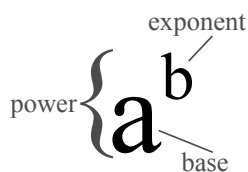(c) $4 \times 5 \div 2 + 7 - 4 \times 4 = ?$

(d) $42 \div 7 \times 3 + 9 \times 2 - 4 = ?$

# Exponents

Recall that multiplication is just repeated addition of a number:

$$4 \times 5 = 4 + 4 + 4 + 4 + 4 \qquad \text{(add 5 groups of 4)}$$

Exponentiation (or using exponents) is based on the same idea, but it is repeated multiplication of a number.



The **base** is the number that you are multiplying over and over again, and the **exponent** is the number of times that the base appears as a factor in the multiplication. The entire expression is called a **power**.

There are also two important rules to remember when dealing with exponents:

> Any number to the exponent 1 is equal to itself.
> Any number (except for 0) to the exponent 0 is equal to 1.

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## Example 2

- $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

- $7^0 = 1$

- $3^2 = 3 \times 3 = 9$

- $5^1 = 5$

- $10^4 = 10 \times 10 \times 10 \times 10 = 10000$

- $189^0 = 1$

- $0^0$ is undefined

- $0^{50} = 0$

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# Brackets

In mathematics, we use brackets to group parts of an expression into sub-expressions. We commonly use parentheses (), but we may also use square brackets [ ], or curly brackets {}.

For example, we can use brackets in the following way:

$$15 - (6 + 1) + 30 \div (3 \times 2)$$

*BEDMAS tells us to evaluate what's in the brackets first.*

Therefore, we get the following:

$$
\begin{aligned}
15 - (6 + 1) + 30 \div (3 \times 2) &= 15 - 7 + 30 \div (3 \times 2) \\
&= 15 - 7 + 30 \div 6 \\
&= 15 - 7 + 5 \\
&= 8 + 5 \\
&= 13
\end{aligned}
$$

If we didn't use brackets, we would get the following:

$$
\begin{aligned}
15 - 6 + 1 + 30 \div 3 \times 2 &= 15 - 6 + 1 + 10 \times 2 \\
&= 15 - 6 + 1 + 20 \\
&= 9 + 1 + 20 \\
&= 10 + 20 \\
&= 30
\end{aligned}
$$

## Nested Brackets

There is no limit on how many sets of brackets we can use in an equation. So you could see an expression that looks like this:

$$(8 - (5 + 1)) \times 3$$

To evaluate an expression like this, we simply follow BEDMAS twice! Once we notice the outer brackets, we realize that we need to first evaluate the sub-expression they contain using BEDMAS. Next, we notice the inner-brackets and then we realize that we need to evaluate that sub-expression first.

A simple rule that summarizes this strategy is:

> When dealing with brackets inside brackets (called **nested brackets**), evaluate what's inside the inner-most brackets first.

Remember that this rule is just BEDMAS. It's nothing new.

Using this rule, our sample expression would be evaluated as follows:

$$(8 - (5 + 1)) \times 3 = (8 - 6) \times 3$$
$$= 2 \times 3$$
$$= 6$$

Now try evaluating the expression without brackets. You should get the same answer.

**Note:** If used effectively, brackets always affect the order in which we do operations in an expression (according to BEDMAS). The final answer may or may not change though.

### Exercise II
Evaluate the following expressions:

(a) $17 - 2^3 + 4 \times 5 = ?$

(b) $72 \div (3 \times 2) - (10 + 1) = ?$

(c) $10 \times (0.4 + 0.3) - 2^2 \div 5^0 = ?$

(d) $10 \times 0.4 + 0.3 - 2^2 \div 5^0 = ?$

(e) $6 + (36 \div 9)^3 \div 2 - 1 = ?$

(f) $(14 - 6) \times ((30 + 5) \div 5) = ?$

(g) $(95 \div 19)^2 + 3 = ?$

(h) $3 \times 7 + 5 - 100 \div 20 \times 4 = ?$

## Solving Equations

As you may already know, we may use symbols (we often use letters) in math to represent an unknown quantity. We call these symbols **variables**. Variables can represent all sorts of different things like the temperature outside, the price of an item, the distance you travelled on your bike, etc. An equation with variables is called an **algebraic equation**.

Try to think about an equation as a balance or a see-saw (teeter-totter). In order for the see-saw to stay balanced, we need both sides to have an equal *weight*. Or in math terms, we need the expressions on both sides of the equals sign to give the same number.

Solving an algebraic equation means finding the value of the variable that *balances* the equation.

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## Notation

From this point on, I will use some new notation when writing mathematical expressions.

- Instead of using the $\times$ sign for multiplication, I will use brackets around each of the factors and write them side by side.

$$\text{ie.} \qquad 12 \times 4 \to (12)(4)$$

  If one of the factors is a variable, we can drop the brackets and just write the factors side by side.

$$\text{ie.} \qquad 12 \times x \to 12x$$

- Instead of using the $\div$ sign for division, I will write the quotient as a fraction with the dividend as the numerator (top) and the divisor as the denominator (bottom):

$$\text{ie.} \qquad 12 \div 4 \to \frac{12}{4}$$

  If the dividend or divisor is a variable, the quotient is still written the same way.

$$\text{ie.} \qquad 12 \div x \to \frac{12}{x} \qquad x \div 4 \to \frac{x}{4}$$

If this is the first time you have thought about fractions as division and it seems strange to you, just think about what a fraction is. It is a whole *divided* into a specific number of parts.

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## Example 3

Consider the algebraic equation below where $x$ represents Adam's age in years:

$$x + 12 = 32$$

The equation tells us that Adam's age *plus* 12 is the same as 32. Or in other words, in 12 years from now, Adam will be 32.

To solve for $x$ in the equation above is to find the value of $x$ that balances the equation.

We need to isolate $x$ on one side of the equation by undoing operations where needed.

To isolate $x$ on the left-hand-side of the equation (LHS), we have to subtract 12 from *both* sides of the equation. We subtract 12 from *both* sides in order to maintain equality. Or you can think of it like this: Since the see-saw is already balanced, if we apply an action to one side, we must apply an equal action to the other in order to keep it balanced.

$$x + 12 \ {\color{red}-12} = 32 \ {\color{red}-12}$$
$$x = 20$$

Therefore, we see that Adam is 20 years old. We have just solved our first algebraic equation!

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## Example 4

Now consider another algebraic equation where $x$ represents the average distance a professional soccer player runs in a game (in km):

$$9x = 90$$

This equation tells us that 9 *times* the average distance run is 90km.

To isolate $x$ in this equation, we can undo the multiplication of $x$ by 9 by dividing both sides of the equation by 9:

$$\frac{9x}{\color{red}9} = \frac{90}{\color{red}9}$$
$$x = 10$$

Thus, the professional soccer player runs an average distance of 10km per game.

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In general, we can develop a method for solving algebraic equations:

1. Simplify the equation as much as possible (ie. evaluate the LHS and RHS expressions as much as possible using BEDMAS)

2. Isolate the variable by *undoing* the appropriate operations with their opposite operations (see the chart on next page). It is important to undo operations in a <u>reverse</u> BEDMAS order.

| Original Operation | | | Opposite Operation |
|---|---|---|---|
| Addition | $+$ | $-$ | Subtraction |
| Subtraction | $-$ | $+$ | Addition |
| Division | $\div$ | $\times$ | Multiplication |
| Multiplication | $\times$ | $\div$ | Division |

Finally, there is also a very important rule that we must always follow when manipulating equations:

> Whatever you do to one side of the equation, you must also do to the other side.

## Exercise III

Solve the following algebraic equations for $x$ using the two-step method:

(a) $x - 4 = 5$

(b) $2x = 22$

(c) $\dfrac{x}{3} = 7$

(d) $6 = x + 1$

(e) $2x - 3 = 5$

(f) $3 = 1 + 2x$

(g) $3x = 2x + 4$

(h) $5x + 1 = 2x + 28$

# Word Problems

Algebra is very useful in solving word problems and it can be helpful when the answer is not obvious.

To solve a word problem with algebra, follow these simple steps:

1. Represent unknown quantities with variable(s).

2. Use relationships given in the problem to set up equation(s) with the variable(s).

3. Solve the equation(s) for the variable(s).

4. Answer the question.

# Example 5

In eight years from now, Sally will be four years younger than three times her current age. How old will she be in two years from now?

**Solution:**

In this problem we are trying to find Sally's age *in two years from now.*

To make things easier, we will first solve for her current age. That will be our unknown quantity to solve for.

Let's apply the 4-step process we just learned:

1. Let $x$ be Sally's current age.

2. In 8 years, Sally will be $x + 8$ years old. 3 times Sally's current age is $3x$. Now we can put together an equation based on the relationship that the problem gives:

$$x + 8 = 3x - 4$$

3. We can solve the equation and find Sally's current age:

$$x + 8 = 3x - 4$$
$$x - x + 8 = 3x - x - 4$$
$$8 = 2x - 4$$
$$8 + 4 = 2x - 4 + 4$$
$$12 = 2x$$
$$\frac{12}{2} = \frac{2x}{2}$$
$$6 = x$$
$$x = 6$$

4. Recall that the question didn't ask for Sally's current age, but it asked for Sally's age in 2 years from now. Therefore, the answer to the question is that Sally will be $x + 2 = 6 + 2 = 8$ years old in two years from now.

# Problem Set

1. Evaluate the following expression:

$$(4)(2+1) - 7 + ((8)(9))^0 - (2)\left(\frac{97+5}{51}\right) = ?$$

2. How much greater is the sum of 6 and 8 than the product of 2 and 6?

3. Victoria has only 5 nickels, 5 dimes, and 5 quarters in her pocket. What is the total amount of money that she has in her pocket?

4. Jimmy goes on iTunes and purchases 8 of his favourite albums with his allowance that he has been saving up. Each album has 8 songs.

    (a) How many songs did Jimmy buy? Express your answer as a power, and then evaluate it.

    (b) If each song is about 4 minutes long, how much time will it take Jimmy to listen to all of his new albums in sequence?

    (c) Jimmy gets home from school at 4pm. Can he listen to all of his new albums before dinner at 7pm?

*5. You are really lucky and win the lottery! As the winner, you can pick one of two possible payment options:

    Option A: Receive 30 million dollars cash!
    Option B: Receive $1 the first day, $2 the next day, $4 the next day, ... (for 25 days).

    (a) Without doing any calculations, what option do you think will yield more money? Explain your reasoning.

    (b) Use exponents to calculate how much money you would receive on the 25th day if you chose Option B.

    (c) Which option yields more money? How do you know?

6. Solve the following equation for $x$ without using the two-step method:

$$(3)(x-5) = (3)(18-5)$$

**Hint:** Think of the equation as a balance/seesaw.

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*7. Use the idea of equivalent fractions to solve for $x$ in the equation below:

$$\frac{(3)(9) - 10}{2} = \frac{34}{x}$$

8. Use the two-step method to solve the following equations for $x$:

(a) $5x + (7)(3) = 39 - x$

(b) $\dfrac{x}{(1 + 2^2)} = (13 - 12 + 11 - 10)\left(\dfrac{30}{6}\right)$

(c) $\left(\dfrac{52 - 34}{6}\right)^3 x + 8 = 40x - (7 + 24)$

Use algebraic equations to solve questions 9 through 15.

9. Seven times a number is 36 more than the number. Find the number.

10. A train consists of an engine and a number of freight cars. The engine is 21 metres in length and each freight car is 11 metres in length. If the length of the train is 362 metres, then what is the number of cars in the train (including the engine)?
    **Hint:** Let $x$ be the number of freight cars in the train.

11. The capacity of the gas tank in Kathy's car is 48 litres. When the tank is one-third full, how many litres of gas must be added to make the tank three-quarters full?

12. The sum of three consecutive integers is 90. What is the largest of the three integers?

13. At the corner store, gumdrops cost 10 cents a piece and sour keys cost 30 cents a piece. If Nathan spent $3.00 at the corner store and bought twice as many gumdrops as he did sour keys, how many sour keys did Nathan buy?

*14. In the diagram, the perimeter of the rectangle is 56. What is the area of the rectangle?

$x$ - 2

$x + 4$
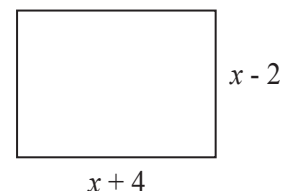
**15. If □□△△ balances ♡, and □□□△ balances ♡△, what balances □? Use algebra to show why your answer is correct. Do not assume the trivial case (all represent 0).
    **Hint:** Use more than one variable and more than one equation.

a. ♡♡       b. □♡       c. △△       d. △

# Answers

**I**

(a) 33      (b) $\dfrac{7}{2}$ or 3.5      (c) 1      (d) 32

**II**

(a) 29      (c) 3      (e) 37      (g) 28

(b) 1      (d) 0.3      (f) 56      (h) 6

**III**

(a) $x = 9$      (c) $x = 21$      (e) $x = 4$      (g) $x = 4$

(b) $x = 11$      (d) $x = 5$      (f) $x = 1$      (h) $x = 9$

**Problem Set**

1. 2

2. 2

3. $2.00

4. (a) $8^2 = 64$
   (b) 256 minutes
   (c) No.

5. (a) Answers may vary.
   (b) $16,777,216
   (c) Option B

6. $x = 18$

7. $x = 4$

8. (a) $x = 3$
   (b) $x = 50$
   (c) $x = 3$

9. $x = 6$

10. 32

11. 20 L

12. 31

13. 6

14. 187

15. c. $\triangle\triangle$

12