UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Grade 7 & 8 Mathematical Circles
November 12/13, 2013
Math Gems

Introduction

In this lesson, we will take a look at some really important formulas and see how we can use geometry and/or algebra to derive them! Then we will also see how to use them to solve some tough problems!

Sum of First n Natural Numbers

Consider This...

You were talking during your math class and as a punishment, your teacher made you sum all the numbers from 1 to 100 together without your calculator. After writing the first ten numbers, you think to yourself that there must be a better way to do this.

Instead, you try to sum the larger numbers first, starting with 100 and working your way down.

This becomes very tedious after a while as well, so you sit there wondering what you could do next to make it easier. Out of boredom, you line up numbers as seen below. You notice something very strange and interesting:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & 96 & 97 & 98 & 99 & 100 \\ 100 & 99 & 98 & 97 & 96 & \dots & 5 & 4 & 3 & 2 & 1 \end{array}$$

Each column sums to 101!

Since there are 100 columns, and each column sums to 101, the total sum of the two rows is (100)(101).

You also know that the sum of each row is equal, which means to find the sum of the numbers from 1 to 100 you just have to perform the following calculation:

$$\frac{(100)(101)}{2} = 5050$$

The famous mathematician Carl Friedrich Gauss used this method to evaluate the sum when he was only in primary school!

Generalizing the Result:

What if you were asked to sum the first n natural numbers together? What would the formula look like? **Hint:** In the question we solved, $n = 100$.

The general formula for the sum of the first n natural numbers, $1 + 2 + 3 + \ldots + n$, is

$$\frac{n(n+1)}{2}$$

Algebraic Proof:

Let $S = 1 + 2 + 3 + \ldots + n$
Write S backwards and line up the sums as done below:

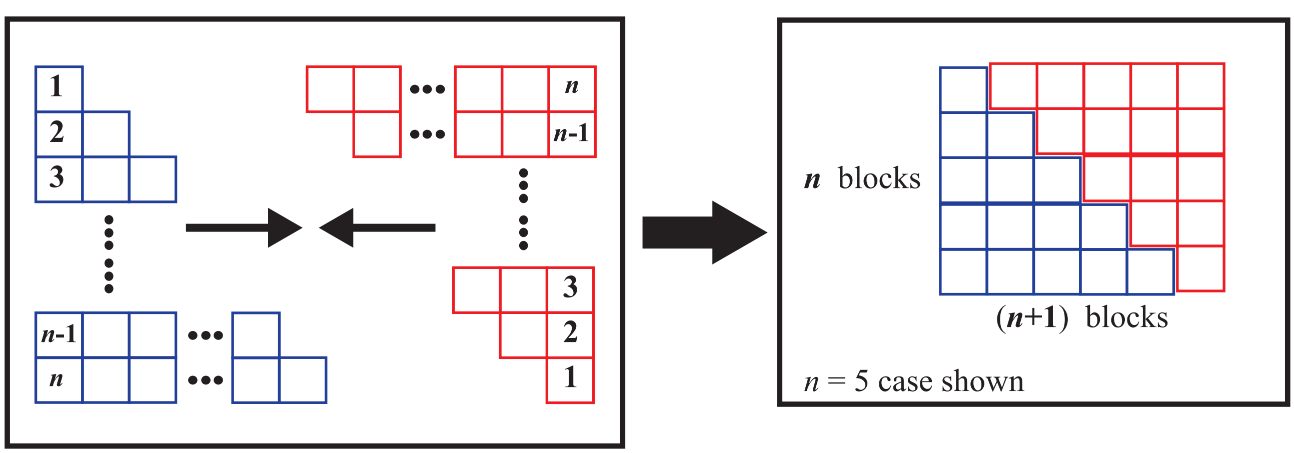
$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & \ldots & + & (n-1) & + & n \\ S & = & n & + & (n-1) & + & \ldots & + & 2 & + & 1 \\ \hline 2S & = & (n+1) & + & (n+1) & + & \ldots & + & (n+1) & + & (n+1) \end{array}$$

There are n columns (terms), so the sum of the two rows is $2S = n(n+1)$.

Therefore, the sum of one of the rows is $S = \dfrac{n(n+1)}{2}$ $\square$

Geometric Proof: (Proof by picture)

Let k unit squares of area 1 represent the number k. That means, one block is the number 1, two blocks is the number 2, three blocks is the number 3, etc.



Lines of blocks (numbers) can be grouped (summed) together as seen above. The sum of the natural numbers 1 to n form a "staircase" shape.

If the "staircase" is duplicated, inverted, and then inter-locked with its original group, an n by $(n+1)$ rectangle is created.

The area of the original "staircase" (ie. the sum of the natural numbers 1 to n) is half the area of the rectangle.

Therefore, the sum of the first n natural numbers is $\dfrac{n(n+1)}{2}$ $\square$

Arithmetic Sequences

We just studied the sum of an arithmetic sequence!

In general, an arithmetic sequence is a sequence of numbers where any two consecutive **terms** (numbers in the sequence) differ by a constant amount.

Examples:

- 1, 2, 3, . . . , 98, 99, 100 (terms differ by 1)
- 2, 4, 6, . . . , 96, 98, 100 (difference of 2)
- 7, 4, 1, -2 . . . -32, -35, -38 (difference of -3)

We know the formula for the sum of the first arithmetic sequence shown. Can we find a formula for the sum of **any** arithmetic sequence?

The Sum of an Arithmetic Sequence (Arithmetic Series)

Let's use the arithmetic sequence $2, 4, 6, \ldots, 96, 98, 100$ as an example.

Notice that if we write out the sum going from left to right, and below it, from right to left, we get

$$\begin{array}{cccccccccccc} S & = & 2 & + & 4 & + & \cdots & + & 98 & + & 100 \\ S & = & 100 & + & 98 & + & \cdots & + & 4 & + & 2 \\ \hline 2S & = & 102 & + & 102 & + & \cdots & + & 102 & + & 102 \end{array}$$

There are 50 even numbers from 2 to 100, so there are 50 terms in this sequence. Therefore,

$$\begin{aligned} 2S &= (50)(102) \\ S &= \frac{(50)(102)}{2} \\ S &= 2550 \end{aligned}$$

Thus, the sum of the arithmetic sequence $2, 4, 6, \ldots, 96, 98, 100$ is $S = 2550$.

What did we do?

The approach we took was the same as the one for the sum from 1 to 100. In fact, it will work for any arithmetic sequence in general.

- We noticed that the sum of all the different pairs were equal
- **Key Point**: This sum will be equal to the sum of the first term and the last term!
- Then we multiplied that sum by the number of terms in the sequence, and divided by 2.

Then the general formula for the sum of an arithmetic sequence is

$$\begin{aligned} \text{S} &= [(\text{Sum of One Pair}) \times (\#\text{ of Terms})] \div 2 \\ &= [(\text{1st Term} + \text{Last Term}) \times (\#\text{ of Terms})] \div 2 \end{aligned}$$

Example

An arithmetic sequence has first term 5, and increases by 2 each time (the common difference is 2). If there are 15 terms in the sequence, find the sum of the terms.

Solution

To use our formula, we need to find the last term of the sequence.

The first term of the sequence is given to be 5.
The second term is $5 + 2 = 7$.
The third term is $7 + 2 = 5 + 2 + 2 = 5 + (2)(2) = 9$
The fourth term is $9 + 2 = 5 + 2 + 2 + 2 = 5 + (3)(2) = 11$

Following this pattern, the fifteenth term of the sequence must be $5 + (14)(2) = 33$.

Then, using our formula, the sum of the arithmetic sequence given is

$$\begin{aligned} S &= [(\text{1st Term} + \text{Last Term}) \times (\#\text{ of Terms})] \div 2 \\ &= [(5 + 33) \times (15)] \div 2 \\ &= [(38) \times 15] \div 2 \\ &= [570] \div 2 \\ &= 285 \end{aligned}$$

The sum of the terms in the arithmetic sequence given is $S = 285$.

Example

You are at a party with 57 people. Everyone shakes hands with everyone else exactly once. How many **unique** handshakes occur?

Solution

We are counting the number of unique handshakes, so if person A shakes hands with person B then we don't also count person B shaking hands with person A as a second handshake.

Since everybody at the party shakes hands with everyone else exactly once, you will shake hands with 56 people (you are the 57th person).

The person who you shook hands with first (call him/her Person 1) will shake hands with 56 people as well. However, one person of those 56 people is you. We have already counted that handshake, so Person 1 makes 55 other unique handshakes.

The second person you shook hands with (Person 2) shakes hands with 56 people as well. However, one person of those 56 people is you, and another one of those 56 is Person 1. Therefore, we count that Person 2 makes 54 other unique handshakes.

Similarly, Person 3 will make 53 other unique handshakes, Person 4 will make 52 other unique handshakes, and so on until Person 56 makes 0 other unique handshakes.

Thus, the total number of unique handshakes that occur is given by the sum of the arithmetic sequence

$$56, 55, 54, 53, \ldots, 3, 2, 1, 0$$

There are 57 terms in the sequence, and we know the first and last terms (56 and 0). So using our formula we get

$$\begin{aligned} S &= [(\text{1st Term} + \text{Last Term}) \times (\text{\# of Terms})] \div 2 \\ &= [(56 + 0) \times (57)] \div 2 \\ &= [3192] \div 2 \\ &= 1596 \end{aligned}$$

Therefore, a total of 1596 unique handshakes occur at the party.

Sum of First n Odd Natural Numbers

Notice something interesting about the arithmetic sequence $1, 3, 5, 7, \ldots$?

$1 = 1 = 1^2$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

$1 + 3 + 5 + 7 + 9 = 25 = 5^2$

$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$

$1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2$

The sum of **consecutive** odd positive integers (starting at 1) is a perfect square! More importantly, the *sum* of the terms is the square of the *number* of terms.

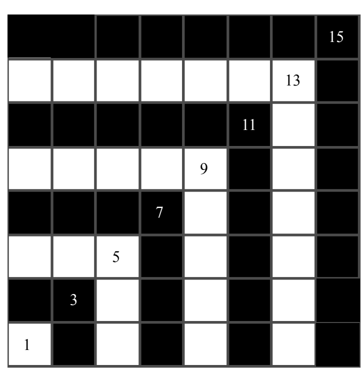
Mathematically, we write this as

$$1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$$

Note: $(2n - 1)$ is the *nth* odd number when we start counting from 1. $2n$ is the *nth* even number when we start counting from 2. This is because, for a given even number, say 20, there are half as many even and odd numbers (10) than the value of that number (20) which are between and including 1 and that given number.

Geometric Proof (Proof by picture.)

Let k unit squares of area 1 represent the number k. That means, one block is the number 1, two blocks is the number 2, three blocks is the number 3, etc.



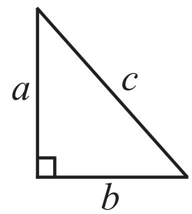
So for a given n, draw a picture like the one above and we will get a square with sides of length n.

Therefore, the sum of the first n odd natural numbers (which is the area of the square) is n^2 $\square$

Pythagorean Theorem

Given a **right-angled** triangle with hypotenuse of length c and legs of length a and b, it is true that

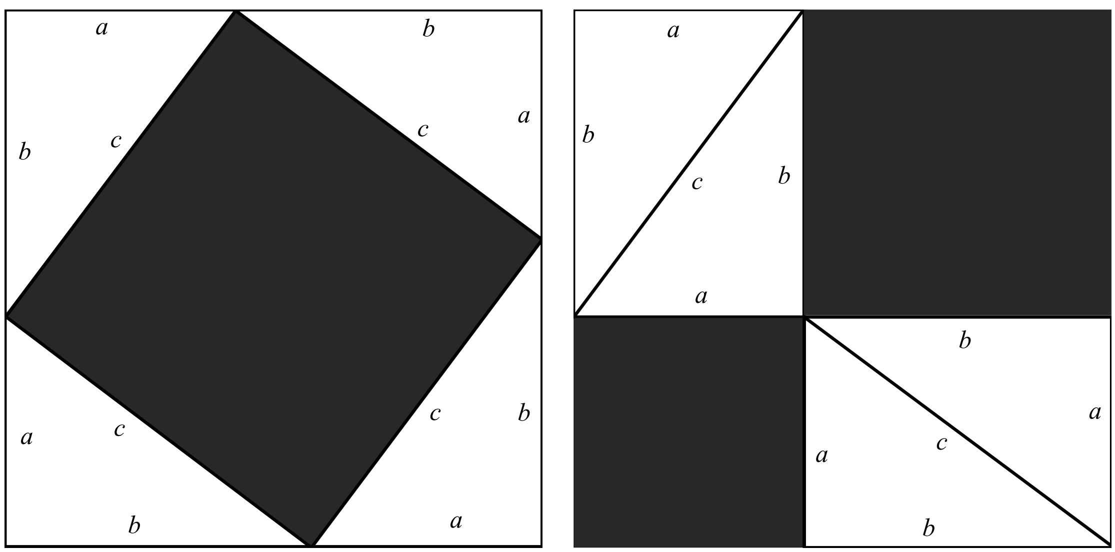
$$\boxed{c^2 = a^2 + b^2}$$



Sketch of Proof

1. In the left diagram, what is the area of the grey square in terms of c? c^2

2. Imagine moving the triangles around until you get the diagram on the right:



3. What is the total area of the dark region? c^2

4. What is the area of the smaller dark square? a^2

5. What is the area of the larger dark square? b^2

6. What can you conclude? Write it out algebraically. $a^2 + b^2 = c^2$

Note: In order to be prove the Pythagorean Theorem formally though, we must prove that the triangles are congruent (you know how to do this) and that the black shaded regions are actually squares (ie. prove sides are equal and interior angles). The full proof is left as an exercise for you.

Why Is Pythagorean Theorem Useful?

The Pythagorean Theorem allows us to determine any side of a right-angled triangle given the other two sides (or some other sufficient information).

Example

Calculate the length of the hypotenuse of the triangle shown above if $a = 4$ cm and $b = 3$ cm.

Solution

The triangle is right-angled so we can use the Pythagorean Theorem:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

Therefore,

$$c = 5 \quad \text{or} \quad c = -5$$

But c is a length, so it cannot be negative. Thus, the hypotenuse of the triangle above is 5 cm.

Pythagorean Triples

The right-angled triangle we just studied with side lengths 3,4,5 is a special triangle. It is the smallest right-angle triangle that has natural number side lengths. We call the set of side lengths $\{3, 4, 5\}$ (any of the same units) a **Pythagorean triple**.

Recall that any two triangles that have all three corresponding sides in proportion must be similar (SSS). Also, similar triangles must have all corresponding angles equal.

Therefore, any scaled copy of the 3,4,5 triangle will also be a right-angled triangle! So we can use Pythagorean Theorem to also check for 90° angles! This is very useful in carpentry! Any set of side lengths that belong to a triangle that is a scaled copy of the 3,4,5 triangle is also called a Pythagorean triple.

There are other groups of Pythagorean triples as well that correspond to right-angled triangles of different shapes. For example, the set $\{5, 12, 13\}$ defines another group of Pythagorean triples.

Can you think of other groups of Pythagorean triples?

Geometric Series

A geometric series is a sum whose terms are related by a **common ratio**. For example, $1 + 2 + 4 + 8 + 16 + 32$ is a geometric series with common ratio $r = 2$ since any term of the series can be divided by 2 to get the previous term in the series (or any term in the series is 2 times as great as the previous term). A geometric series can have a finite or infinite number of terms.

Why is it useful?

Remember this problem? I have just changed the numbers for demonstrative reasons.

Harry the mathematical turtle is standing 2 *metre from a wall. Because Harry gets tired very easily, it takes him one hour every time to walk halfway toward the wall. This means that it takes him an hour to walk* 1 *metres, then an hour to walk* 0.5 *metres, and so on. Will he ever reach the wall? And if he does, how long will it take him?*

Ignoring any time constraints, we can show that Harry actually reaches the wall as follows:

The total distance Harry travels (in metres) is given by the geometric series

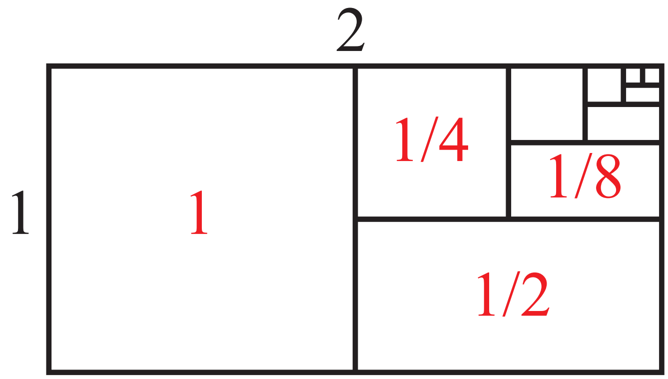
$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

If we let S represent the sum of this series, we can do the following trick to find a value for S:

$$\begin{aligned} 2S &= 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ (-)\ S &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \\ \hline 2S - S &= 2 \end{aligned}$$

Therefore, $S = 2$. So Harry walks a total distance of 2 metres without any time constraint. Of course, it would take him an infinite amount of time to actually travel this distance, so the debate still remains as to whether he actually reaches the wall. This may only happen if Harry is immortal!

You can make a geometric argument for why this result is true by subdividing a 2 by 1 rectangle repeatedly as shown below. Each inscribed rectangle corresponds to a term of the geometric series that models the distance travelled by Harry.



Since there are an infinite amount of terms in the series, we can keep making more subdivisions within the rectangle. Therefore, the sum will never exceed the value 2 (the area of the rectangle) but it will always be increasing with the addition of every term. Thus, after adding an infinite amount of terms, the sum must be equal to 2.

Finite Geometric Series

A finite geometric series is defined by three parameters:

1. A starting value (first term), a
2. A common ratio, r
3. The number of terms in the series, n

A geometric series with these parameters has the form

$$a + ar + ar^2 + \ldots + ar^{n-1}$$

Examples:

- $4 + 12 + 36 + 108 + 324$ $\quad (a = 4,\ r = 3,\ n = 5)$
- $7 + \frac{7}{5} + \frac{7}{25} + \frac{7}{125}$ $\quad (a = 7,\ r = \frac{1}{5},\ n = 4)$
- $\frac{3}{8} + \frac{3}{4} + \frac{3}{2} + 3 + 6 + 12 + 24$ $\quad (a = \frac{3}{8},\ r = 2,\ n = 7)$

We won't prove this result (since it requires some advanced math), but the sum of a geometric series of the above form (general form) is given by the formula

$$S = a\left(\frac{1 - r^n}{1 - r}\right)$$

where S just represents the value of the sum.

Problem Set

*difficult **challenge

The questions require the use of different formulas or tricks you learned in the lesson. Try to figure out which tools to use for each problem.

1. If Johnny walks 600 m west and 800 m north, how far away from his starting point is he? Give your answer is in metres.

2. (a) What is the sum of the integers from 1 to 187?

   (b) What is the sum of the integers from 1 to 50?

   (c) What is the sum of the integers from 51 to 187?

3. Evaluate the sum $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots + \frac{271}{2}$

4. If the sum of the lengths of the legs of a right-angled triangle is 21 cm, and if the hypotenuse is 15 cm, then what are the lengths of the triangles legs (answer in cm)? **Hint:** Use Pythagorean triples. There is only one possible set of side lengths.

5. Julia delivered newspapers last year (for 52 weeks). The first week, she earned \$30 for the week. Every week, Julia was given a \$1 raise on her weekly salary. How much money did Julia make over the entire year?

6. For this question round your answers to the nearest tenth of a cm^2.

   (a) Find the area of an equilateral triangle with 10 cm sides.

   (b) Find the area of a regular hexagon with 4 cm sides. **Note:** A regular polygon has all sides equal and all interior angles equal.

7. The world has been taken over by aliens with extremely long life spans (like 100's of years) and fast breeding rates! Every night at midnight, each breeding alien spawns three new aliens. And once an alien is spawned, it will breed the next day! Thankfully, each alien can only breed once in its lifetime. An initial population of 100 breeding aliens landed on planet Earth on Thursday. (Assume that no other aliens land on Earth afterwards)

   (a) If today is Tuesday, what is the current alien population on Earth?

   (b) If the current growth rate stays the same, then what will be the alien population on Earth in 2 weeks from today (Tuesday)? **Hint:** There is a formula for this!

8. * **Arithmetic Series.**

   An arithmetic series is a sum where the terms in the sum are related by a **common difference**. For example, $3 + 7 + 11 + 15 + 19$ is an arithmetic series with a common difference of $d = 4$.

In general, we can define a finite arithmetic series with a common difference d, a starting value (first term) a, and a number of terms n. That is,

$$a + (a + d) + (a + 2d) + \ldots + (a + (n - 1)d)$$

is an arithmetic series of n terms, starting at a, with a common difference of d.

(a) Using the same algebraic strategy that we used in the lesson to derive the formula for the sum of the first n natural numbers, try to derive the formula for the sum of a general arithmetic series:

$$S = \left(\frac{n}{2}\right)(2a + (n - 1)d)$$

where S just represents the value of the sum. **Hint:** You have seen this formula before. It's just written differently now.

(b) Use the formula given in part (a) to evaluate the sum of the first 10 terms of the infinite arithmetic series $5 + 13 + 21 + 29 + \ldots$

9. * An important algebra fact that you will learn in high school is that if you have two whole numbers a and b, then $(a + b)^2 = a^2 + 2ab + b^2$.

(Algebraic) Proof:

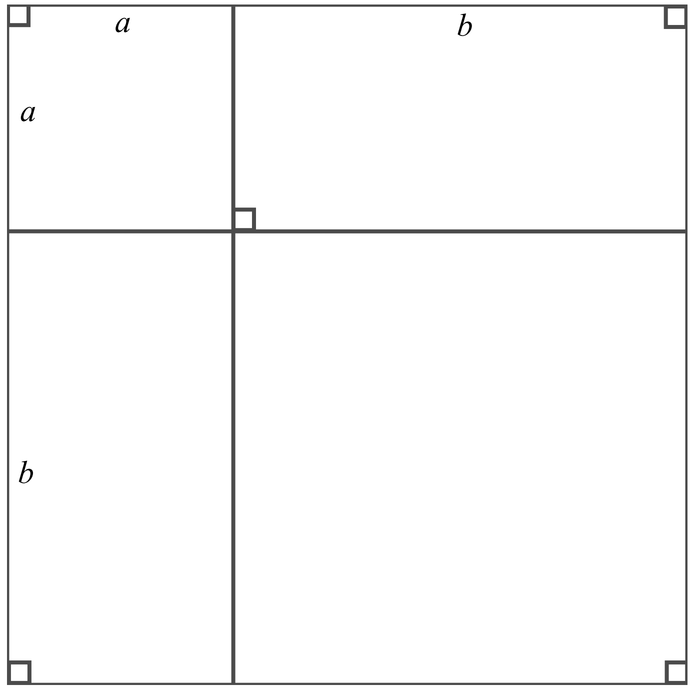
$$(a + b)^2 = (a + b)(a + b)$$

Let $x = (a + b)$. Then

$$\begin{aligned}
(a + b)^2 &= (a + b)(a + b) \\
&= x(a + b) \\
&= (x)(a) + (x)(b) && \text{(Distributive Property)} \\
&= (a + b)(a) + (a + b)(b) && \text{(Substitute } x = (a + b)\text{)} \\
&= && \text{(Distributive Property again)} \\
&= \\
&= a^2 + 2ab + b^2 && (ab = ba)
\end{aligned}$$

(Fill in the remaining steps of the algebraic proof.)

Can you prove the same result by using geometrical arguments and the following diagram?



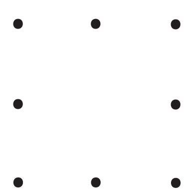
10. ** Solve for x given that

$$25 = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$$

Hint: Use a similar strategy to the algebraic one we used to evaluate the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

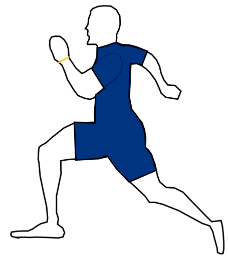
11. Find the sum of the integers from 5 to 34.

12. * In the following diagram, how many unique ways can you join two points?



13. Find the sum of this sequence: 1, 4, 7, 10, 11, $\dots$, 73

14. Hector and Achilles are having a race. After 1 s, the distance between the two of them increases by 1 cm. After 2 s, the distance between them increases by 4 cm; after 3 s, 7 cm; after 4 s, 10 cm, and so on, with the distance increasing by 3 cm every second. How far apart are they after 25 s?



15. (a) In an arithmetic sequence, the first term is 13 and the terms go up by 5. Find the last term if there are 21 terms in total.

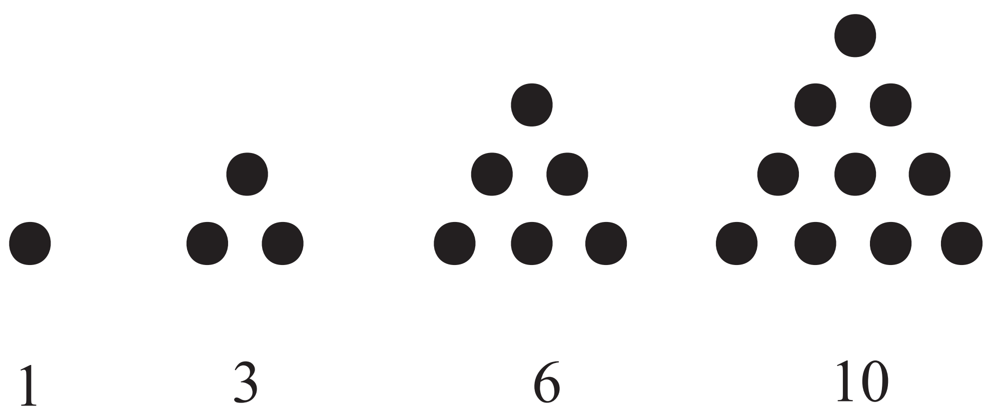
    (b) In an arithmetic sequence, the first term is 4 and the terms go up by 3. If the last term in the sequence is 49, how many terms are in the sequence?

    (c) In an arithmetic sequence, the last term is -3, and the terms decrease by 5 each time. Find the first term if there are 20 terms in total.

16. Sebastian just had knee surgery. His trainer tells him to resume jogging slowly: 12 minutes the first week, 18 minutes the second week, and so on, increasing the number of minutes by 6 every week. How long will it be before Sebastian is jogging 78 minutes a week?

17. The sum of the interior angles of a triangle is 180°; a square 360°; a pentagon 540°. Assuming this pattern continues, what is the sum of the interior angles of a polygon with 14 sides?

18. The sequence of triangular numbers counts the number of objects that can be used to form an equilateral triangle. (The first 4 are shown below)



    What is the 23rd triangular number?

19. * A python starts at the base of a smooth, cylindrical tree trunk and winds itself tightly 4 times around the trunk until its head rests at the top. The tree trunk has a circumference of 3 feet, and is 9 feet tall. How long is the python?

20. * The 7th term in an arithmetic sequence is 23, and the 12th term is 38. Find the first term and the common difference.

21. ** The numbers $x + 1$, $3x - 1$, and $4x + 1$ are consecutive terms in an arithmetic sequence. Find x.