UNIVERSITY OF WATERLOO

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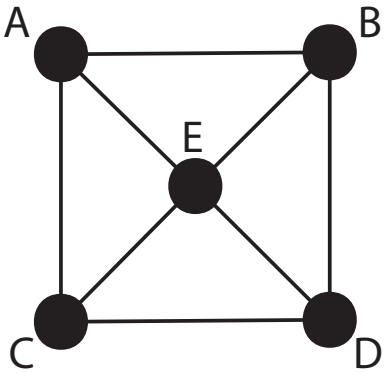
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

March 4/5, 2014

Graph Theory I



Source: `http://world.mathigon.org/Graph_Theory`

Consider the picture and graphical representation above. There are 5 islands and 8 bridges connecting them. Can you find a way to cross all the bridges exactly once? You can start and end anywhere, but remember that you must cross every bridge, and you can't cross the same bridge more than once. Note: This is not possible since there are more than two islands with an odd number of bridges. Think about how this makes it impossible.

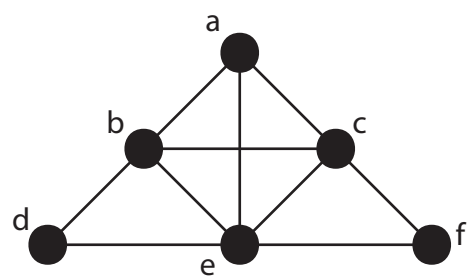
Terminology

- A **vertex** (plural: **vertices**) is a point.
- An **edge** is a line that joins two vertices.
- Two vertices are **adjacent** if there is an edge joining them.
- A vertex and edge are **incident** if they are joined.
- A **walk** is a sequence of vertices where each vertex is adjacent to the vertex before and after it.
  - Think about walking from one vertex to another (on edges) and keeping track of all the vertices you pass on the way.
- A **path** is a walk that doesn't repeat vertices.
- A **cycle** is a path that begins and ends at the same vertex.

Notation

- A **vertex** is usually labelled with a letter, number, or other title. (ex. A)
- An **edge** is named in the format $\{-,-\}$, where " $-$ " is replaced by the vertices it is incident to. (ex. $\{A, B\}$)
- A **graph** is usually named G unless it is special or well-known (we will see examples of these in the problem set). To distinguish graphs, a subscript can be used. (ex. G_1)
- A **walk** can be written as "$vertex, edge, vertex...$" or as "$vertex, vertex, vertex...$". We will use the vertex only notation (ex. $A \to B \to E \to A \to C$)
- A **path** can be written in the same way as a walk. (ex. $A \to B \to E \to C \to D$)

Exercise 1



G

1. Use the graph above to answer the following questions

   (a) List the vertices in G. a, b, c, d, e, f

   (b) List the edges in G incident to b. {a,b},{b,d}, {b,e},{b,c}

   (c) What is the difference between the terms "incident" and "adjacent"? Incident is used when talking about an edge and a vertex. Adjacent is used when talking about two vertices.

   (d) Vertex a is adjacent to vertex e.

   (e) Vertex d is incident to edge $\{d, e\}$.

   (f) Create a walk from b to f. Answers will vary. Example: b,e,f

   (g) Is your walk from part (e) also a path? If not, find one that is and explain why it is a path. If it is already a path, explain why and find a walk that isn't a path. Answers will vary. A path should have no repeating vertices.

   (h) Find a cycle in G. Answers will vary. Example: a,b,e,c,a

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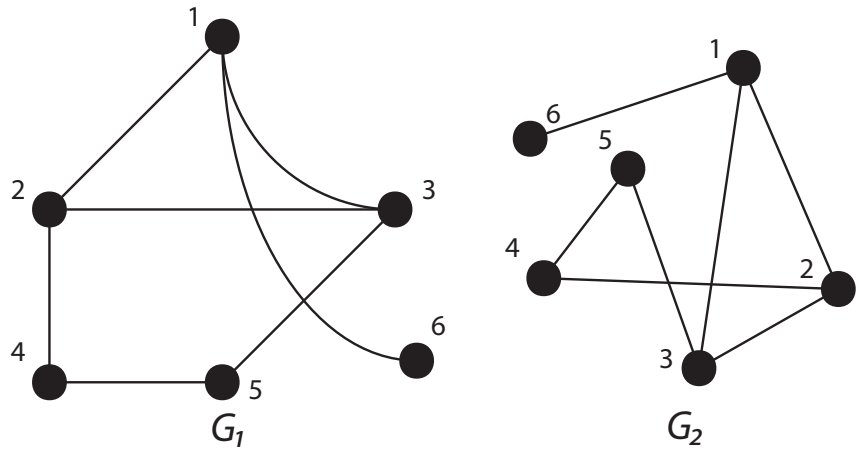
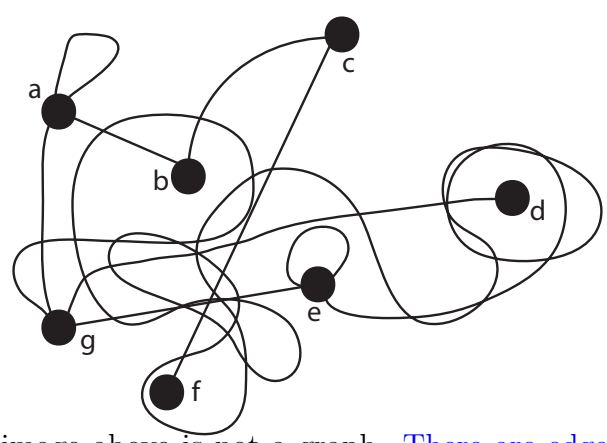
Rules

These are the rules that we will be using. Different types of graphs and different parts of graph theory have different rules.



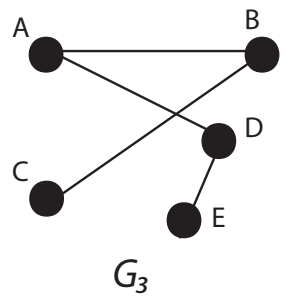
- Edges **can** be curved or straight (or any other shape).
- Edges **can** cross.
- The placement of vertices **doesn't** matter. Only the connections are important.
  - For example, G_1 is the same as G_2 since all the connections are the same.
- Two edges **can't** connect the same vertices. (ie. There can only be a **maximum** of one edge between any two vertices.)
- An edge **can not** connect a vertex to itself.

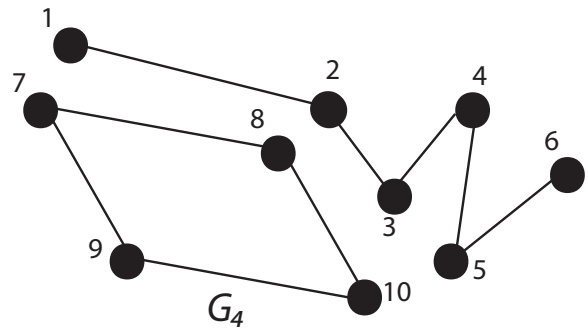
Exercise 2



1. Explain why the image above is not a graph. There are edges connecting vertices to themselves (a and e, and there are two edges connecting the same vertices (e and g)

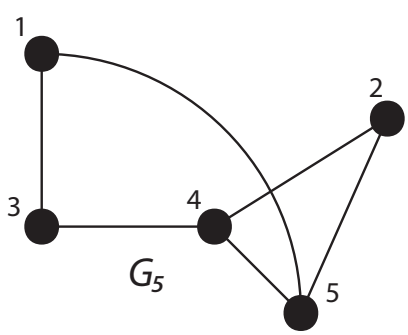
Connectedness

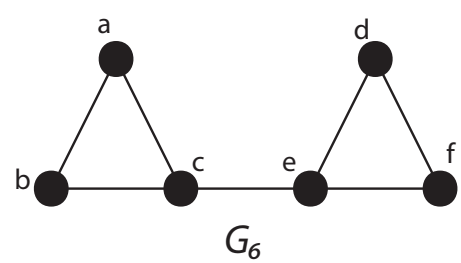




- A graph is **connected** if all the vertices are joined together (either directly or through other vertices). G_3 is connected.
- A graph is **not connected** if you can find two vertices that are not joined together (either directly or through other vertices). G_4 is not connected.
- A **component** is a group of connected vertices. For example, G_3 has 1 component, but G_4 has 2 components.
- A **bridge** is an edge that joins two components into one. In other words, when a bridge is removed 1 component is split into 2.

Exercise 3





(a) Are G_5 and G_6 connected? Yes, both G_5 and G_6 are connected

(b) How many bridges does G_5 have? What about G_6? G_5 doesn't have any bridges. G_6 has 1 bridge.

(c) If you remove all bridges are G_5 and G_6 still connected? Since G_5 doesn't have any bridges, it is still connected. G_6 is not connected.

(d) If you remove all bridges how many components are there? There are 3 components. G_5 is one component and G_6 has 2 components

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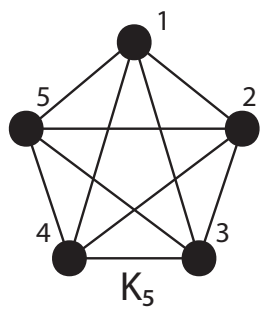
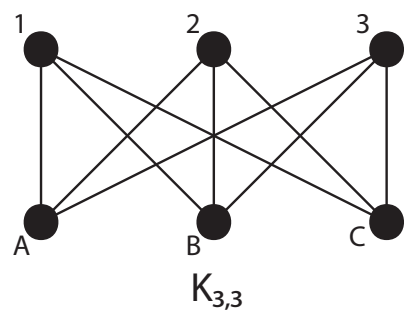
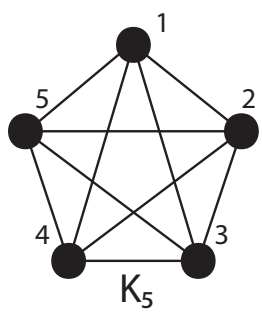
Planar Graphs

- A **planar graph** is a graph that can be drawn without any edges crossing. When a planar graph is drawn in its planar form (without any edges crossing), we call it a **planar embedding**.
- An enclosed area bounded by vertices and edges in a planar embedding is called a **face**.
- It may seem like a planar embedding is always possible, but there are some graphs that are impossible to draw without any edges crossing. We call these graphs **non-planar**. There are two notable non-planar graphs we will look at: K_5 and $K_{3,3}$.



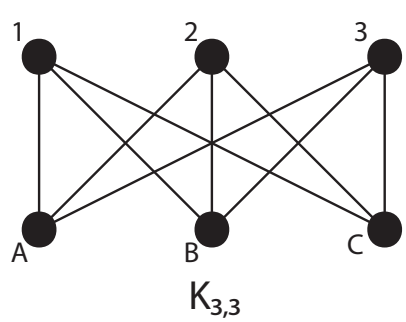


K_5- Complete Graph



- A **complete graph** is a graph where all the vertices are joined to each other.
- A complete graph is called K_n where n is the number of vertices in the graph.
- A complete graph has all possible edges. This means that you can not draw any more edges.

$K_{3,3}$- Complete Bipartite Graph



- A **bipartite graph** is a graph where the vertices can be split into two groups so that none of the vertices in a group are adjacent to each other. In other words, all the vertices in a group are not adjacent.
- A bipartite graph is called $K_{m,n}$ where m and n are the number of vertices in each group.

Exercise 4

1. Although it is not possible to draw a planar embedding of K_5 and $K_{3,3}$, we can rearrange the vertices to make them close to planar.

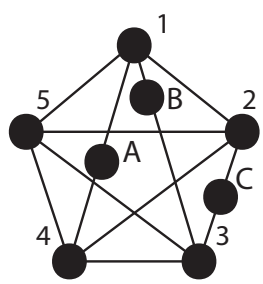
   (a) The **crossing number** of a graph is the least number of edges crossing. Find the crossing numbers of K_5 and $K_{3,3}$. Draw this. The crossing number of both graphs is 1. Drawings may vary

   (b) Explain why the drawing from part (a) can't become planar. Explanations may vary. Students should realise that there vertices will be surrounded by edges and there is no way to draw the remaining edge without crossing, regardless of the order the edges are drawn in.

   (c) If you add more edges and/or use vertices to split existing edges (you can't add a vertex where two edges cross) to either of the graphs, can they become planar? No, the graph will remain non-planar. This is discussed in more detail below.

- A **subgraph** is any graph within a graph. In other words, it is a part of a graph. You can remove any edges or vertices from a graph to make a subgraph.

- A **subdivision** of a graph is a graph that has vertices added to divide an existing edge. This is a subdivision of K_5:



K_5 subdivision

Theorem. *A graph is non-planar if and only if there is a subdivision of K_5 or $K_{3,3}$.*

<u>Note:</u> "If and only if" means that the first statement is true if the second is true, and the second is true if the first is true.

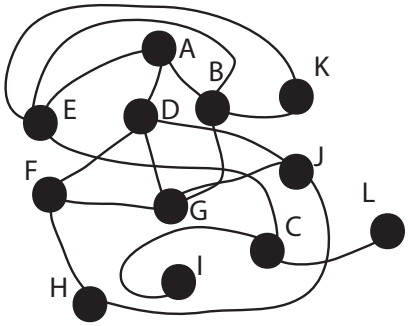
This means that if a graph has a subgraph or subdivision of K_5 or $K_{3,3}$ then it is non-planar.

Problem Set

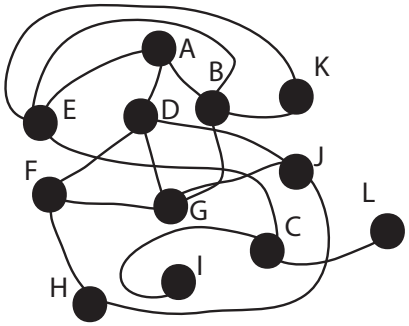
"*" indicates challenge question

1. Trace the following walks on the graph below. For each one, state whether it is a path? How do you know?

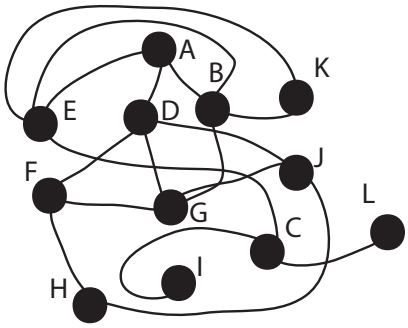
   (a) L-C-E-A-B-A-D

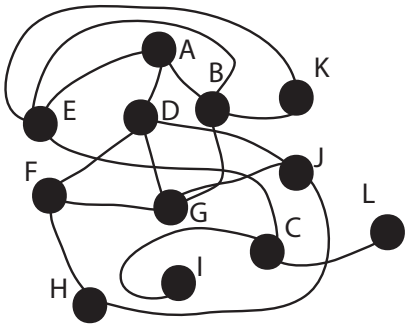
   (b) H-F-G-J-D-A

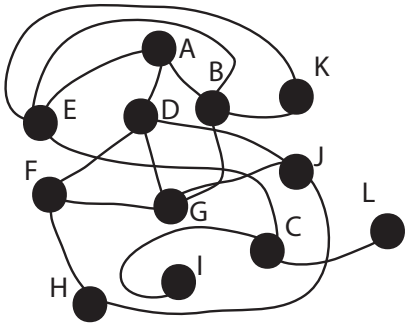
   (c) F-D-A-B-K-E-A

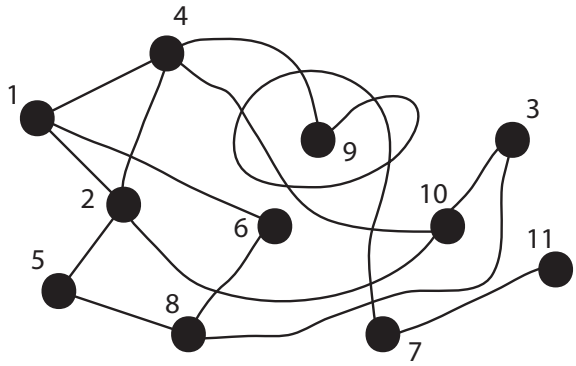
   (d) F-G-J-H

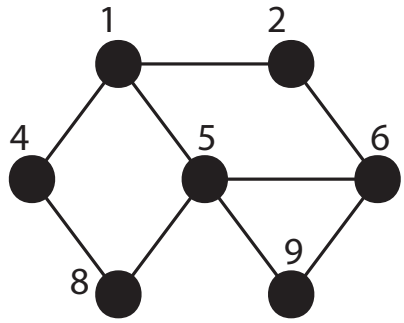
   (e) D-A-E-B-K-E-C

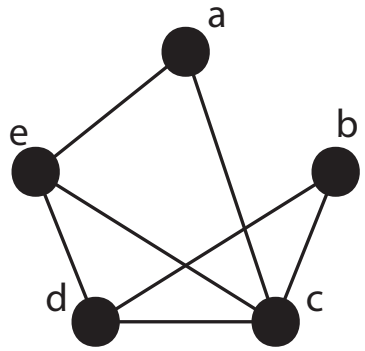
2. Find 10 walks from 1 to 4 in the following graph. How many of these walks are paths? How many possible paths are there between 1 and 4. (Hint: You may want to redraw the graph)



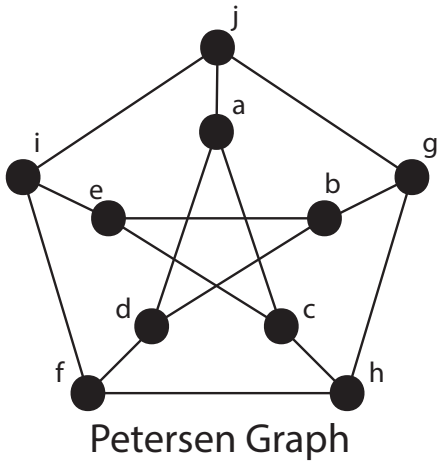
3. Find all the cycles in the following graph.



4. Which edges should be added to the graph below to make it K_5? Write the names of the edges and then draw them on the graph.

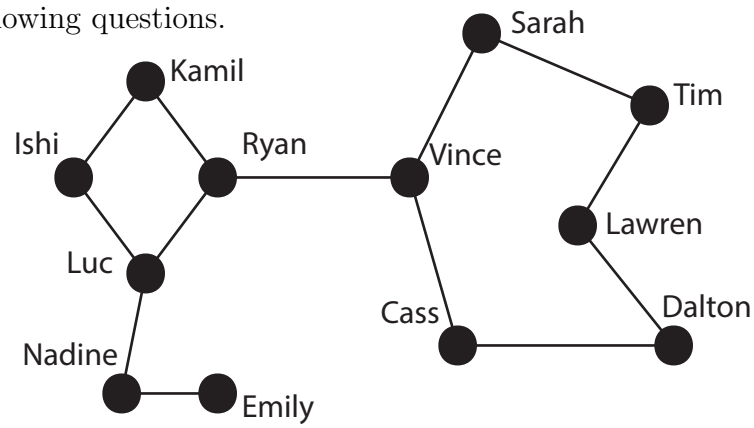


5. A k**-regular graph** is a graph with k edges incident to each vertex. An example of this is the Petersen Graph which is a type of 3-regular graph.



Luc, Ryan, Vince, Emily, and Nadine go to a party. When they get there they want to shake hands but they only have time to shake two other people's hands. Draw two different graphs to show how this can happen. How many handshakes are there? Do the rules for drawing a graph make sense in this situation?

6. Sarah has a letter that she wants to give to Emily. But they won't see each other because they are in different cities. Below is a map of people who will see each other. Answer the following questions.



(a) Which people must see each other in order for Sarah to get the letter to Emily? In other words, if these people don't meet then it is impossible for Sarah to get the letter to Emily.

(b) What do these vertices have in common?

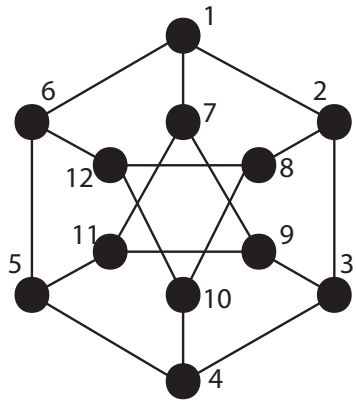
(c) If Ryan and Vince don't meet, what changes about the graph (other than removing an edge)?

(d) Vince lives far from Sarah so she doesn't want to give him the letter. What is the fastest way now?

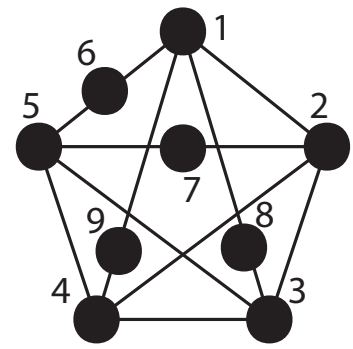
(e) Dalton has a letter that he wants to give to Ishi. What is the fastest way for him to do this?

7. Are the following graphs planar? If so, draw a planar representation. If not, explain why. Hint: Only one of these are planar. Try finding K_5 or $K_{3,3}$ subdivisions.

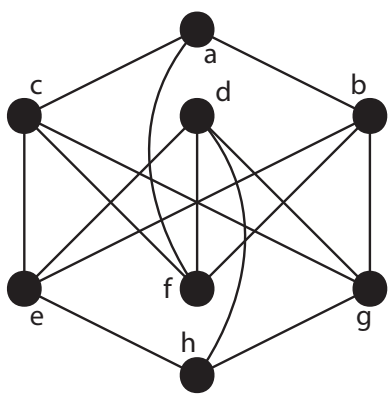
(a)



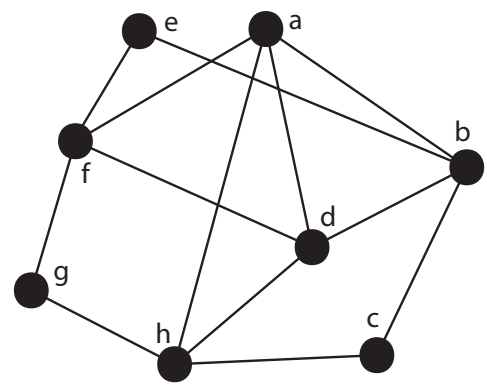
(b)



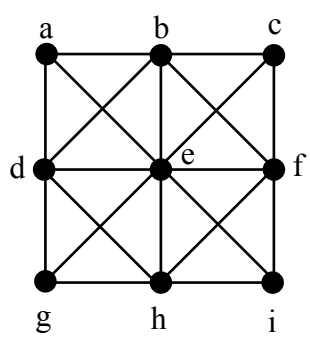
(c)



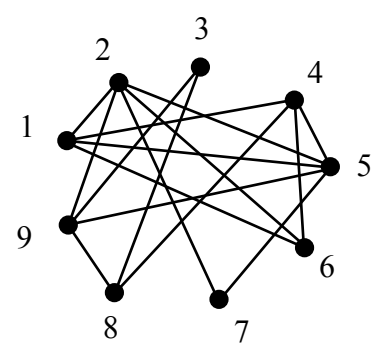
(d)



8. Redraw each graph to show it is planar.



(a)



(b)

9. A word graph has words as the vertices. Two words are adjacent if they differ by 1 letter. Create a connected word graph that contains the following words: log, top, eat, mud and rag.

10. Below is a map of South America. Create a graph with the vertices representing countries and the edges joining those countries which share a land border. So Chile and Peru are adjacent, but Peru and Uruguay are not. Use your graph to answer the questions below.

Courtesy of abcteach.com

(a) When trading goods over land, a $100 tax is paid to each country which the goods travel through. So if Columbia sells coffee to Venezuela, the least amount of tax paid is $100.

i. What is the least amount of tax paid on wool shipped from Ecuador to Paraguay?

ii. What country can trade with the most others for exactly $100?

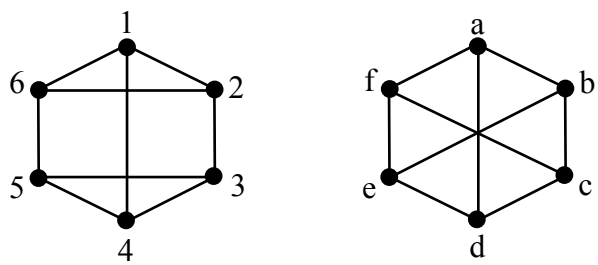
iii. Brazil raises it's taxes to $400. What is the cheapest amount of taxes paid to ship Venezuelan oil to Paraguay? What about French Guiana to Bolivia?

iv. To encourage trade, Peru will not tax any goods going though the country. Chile and Suriname want to trade lumber and wheat. What route will result in the least amount of taxes? How much will they pay?

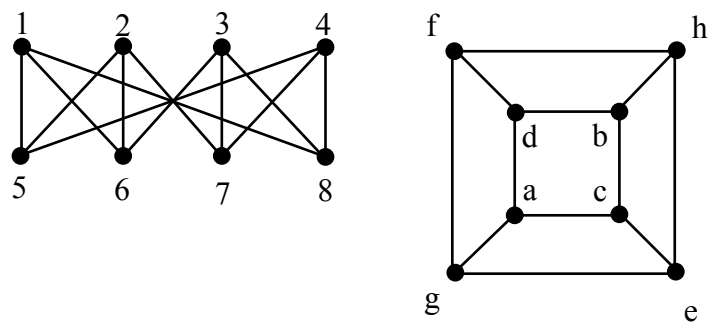
(b) Kevin wants to visit every country in South America on his vacation. He doesn't want to visit any country more than once since crossing the border can take a long time! He will fly into Lima, Peru to begin his trip and fly out of Montevideo, Uruguay. In what order should he visit each country?

11. *Two graphs are **isomorphic** if they are the same graph drawn in a different way. This means that all the connections are the same, and there are exactly the same number of vertices, but the vertices may have different names and the graphs may look different. Determine if the following pairs of graphs are isomorphic. If so, show which vertices match, or explain why not.

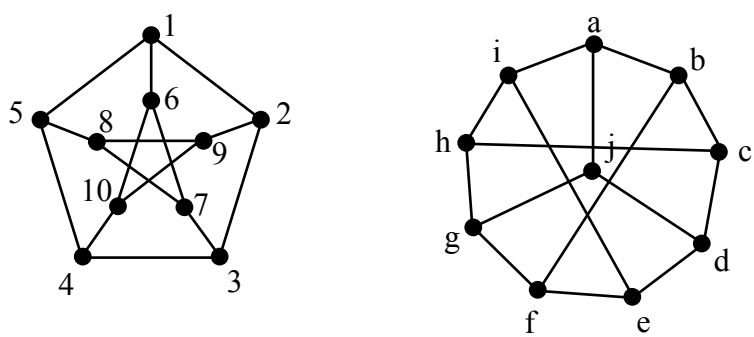
(a)



(b)



(c)



12. Euler's Oil is a gas station chain in the country of Mathisfun. The CEO of the company, Leonhard, wants to visit all his gas stations in the country to ensure that each one is stocked with enough fuel. However, he doesn't want to visit any town more than once as he is a very busy man. Suggest a route Leonhard could take from the head office in the capital Mathtopia, through each town exactly once, and return to the capital. Note that despite the intersection of three highways in the centre of the country, there is no way to change from one road to another (i.e. there is no direct road from Mathtopia to Gausston, etc.)