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Grade 6 Math Circles

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Probability

Probability is the likelihood of an event occurring. Understanding it and knowing how to use it to predict outcomes can be very helpful in areas like:

- **Weather Forecast**
- **Major League Baseball**
- **Lottery (any game of chance)**
- **Medical Operations**

Theoretical Probability - the number of ways an event can occur, divided by the total number of possibilities

Experimental Probability - an estimate of the likelihood of an event occurring based on the collection of data over a long period of time.

Theoretical probability is useful, because it can be calculated with an equation before any experiments need to be done. The probability of any event **A** occurring is:

$$\text{P(A)} = \frac{number\ of\ ways\ \mathbf{A}\ can\ occur}{total\ number\ of\ possible\ outcomes}$$

Before we can fully understand this equation, we must introduce a definition.

Sample Space

The Sample Space of a given activity is the set of all possible outcomes that can happen when doing that activity. Sample Spaces vary in size and there are several procedures in finding how many possibilities are included in these spaces.

Example: A fair coin is tossed 3 times, with each toss being recorded.

1. What is the sample space and the size of this sample space?
2. Let **A** be the event that exactly 2 heads occur. How many different ways can **A** occur?
3. What is the probability that event **A** will occur?

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1. **{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}**

   There are 8 different possible outcomes, so the size of the sample space is 8.

2. Looking at the Sample Space above, we have the following occurrences with exactly 2 heads:

   {HHT, HTH, THH} Thus A can occur in 3 different ways.

3. Using our discoveries in the above questions, the probability that event **A** will occur is:

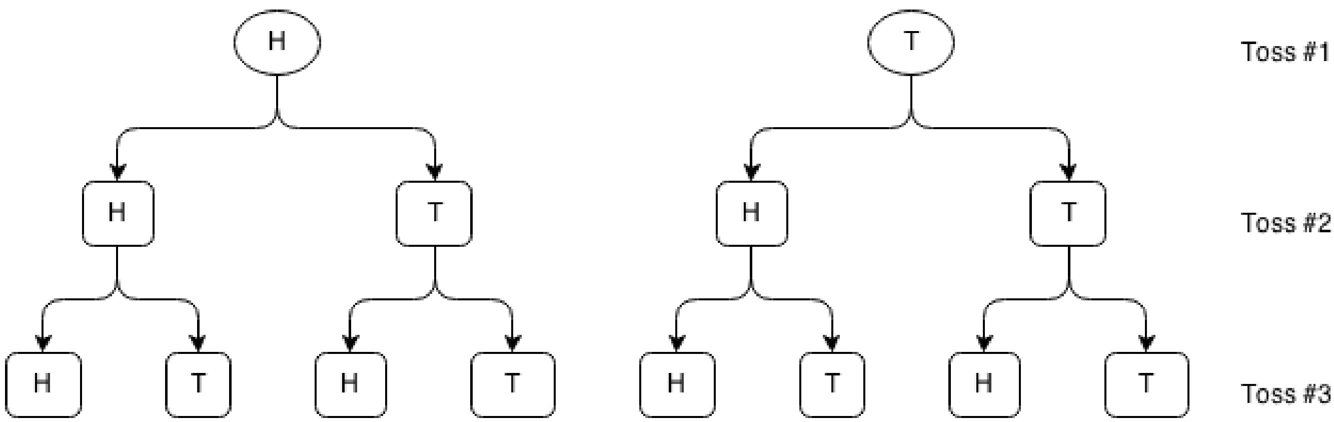
$$\mathrm{P(A)} = \frac{number\ \ of\ \ ways\ \ \mathbf{A}\ \ can\ \ occur}{total\ \ number\ \ of\ \ possible\ \ outcomes} = \frac{3}{8} = 37.5\%$$

To find the probability of an event occurring, find the size of the sample space and the amount of times the event occurs within that sample space. Then, divide this number of occurrences by the size of the sample space.

Finding the Sample Space

Tree Diagrams

In some cases we will find that Sample Spaces are not always easy to form. It is useful to have a simple method of doing this, and that is where a **tree diagram** comes in handy. In the example above, we could have found the Sample Space as follows:



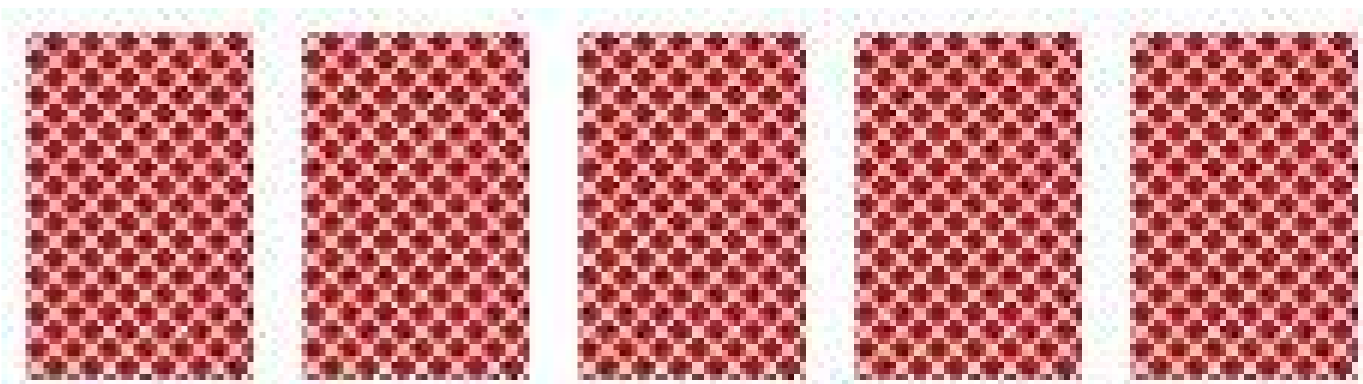
To form each 3 coin flip combination, we start from the top of each tree then trace down each route separately.

Counting - Using Math

Sometimes there are instances when drawing a tree diagram becomes difficult because of the number of outcomes. For example, let's look at some probabilities that arise when playing with a deck of cards.

Example: A dealer randomly selects 5 cards from a fair deck of cards. Calculate the number of possibilities within the Sample Space and the number of occurrences of event **B**, such that at least 3 of the 5 cards selected are hearts.

Using a tree diagram to find out the size of the Sample Space would take forever in this case; so we must use a different method called, "Counting". Luckily enough, we can use a picture and math to solve this problem.



Notice that a deck of cards contains 52 cards. This implies that the first card has 52 different possibilities. Once that card is shown, there are then 51 cards left in the deck, implying that for each and every first card we select, there are 51 possibilities for the second card. Applying this logic down to the 5th card, mathematically, the number of possibilities in our Sample Space is:

$$52 * 51 * 50 * 49 * 48 = 311{,}875{,}200$$

We then use this same logic to count the number of occurrences of event **B** within our Sample Space. Notice that there are 13 hearts in a deck of cards and we only need 3:

$$13 * 12 * 11 * 49 * 48 = 4{,}036{,}032$$

Multiple Events

A sample space will always have the same number of possibilities within it, but we can define as many events as we want inside or outside of that sample space. This is useful when we are looking at examples where several probabilities come into play. Let's introduce some more definitions:

1. **Independence** - Two events **A** and **B** are said to be independent of each other if the result in the outcome of one has no effect on the other.

   - If **A** is the event that the last coin flip in a series of 3 is heads, and **B** is the event that the first coin flip in a series of 3 is tails, are events **A** and **B** independent?

     Whether or not a tails lands for the first flip has no effect on the probability that a heads will land for the last flip. Therefore, these two events are independent.

2. **Intersections** - The probability of an intersection of events **A** and **B**, written as $P(A \cap B)$, is the probability that events **A AND B** will occur at the same time. If the two events are independent, then $P(A \cap B) =$ P(A)*P(B).

   - From the example above, write in words what $P(A \cap B)$ means.
   - Find $P(A \cap B)$

   Find $P(A \cap B)$ is the same as saying, "Find the probability that the last coin flip in a series of 3 is heads **AND** the first coin flip in a series of 3 is tails".

   One way of doing this is writing down the sample space and just looking for when both of these events occur at the same time:

$$S = \{\text{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}\}$$

$$\text{So } P(A \cap B) = \frac{2}{8} = \frac{1}{4} = 0.25 = 25\%$$

   Alternatively, we can use counting and our definition of independence to solve this probability:

Let's find the number of occurrences for event **A**. We know that both of the first two flips can be either heads or a tails, so there are 2 outcomes we can have for both the first flip and the second flip. But for the last flip, we need it to be heads, so there is only 1 outcome we can have. Thus there are,

2 * 2 * 1 = 4 occurrences for event **A**.

Similarly for event **B**, we need the first flip to be tails, so there is only 1 outcome we can have for the first flip. The other 2 flips can be either heads or tails. Thus there are,

1 * 2 * 2 = 4 occurrences for event **B**

We know the size of our sample space is 8, so

$$P(A) = \frac{4}{8} = \frac{1}{2} = P(B)$$

$$\text{Then, } P(A \cap B) = P(A) * P(B) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4} = 0.25 = 25\%$$

3. **Unions** - The probability of a union of events **A** and **B**, written as $P(A \cup B)$, is the probability that events **A OR B** will occur. It can be calculated as follows:

$P(A \cup B)$ = P(A) + P(B) - $P(A \cap B)$.

We subtract the intersection once to ensure we do not double count.

- From the example above, write in words what $P(A \cup B)$ means.
- Find $P(A \cup B)$.

Find $P(A \cup B)$ is the same as saying, "Find the probability that the last coin flip in a series of 3 is heads **OR** the first coin flip in a series of 3 is tails"

Let's find the union probability by listing out the occurrences for events **A** and **B** then adding them together:

Our sample space is the following:

$$S = \{\text{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}\}$$

Looking at the sample space, the occurrences that work for event **A** are:

$$A = \{\text{HHH, HTH, TTH, THH}\}$$

The occurrences that work for event **B** are:

$$B = \{\text{TTT, TTH, THT, THH}\}$$

Let's unite these events by combining the above two lists of occurrences:

$$A \cup B = \{\text{HHH, HTH, TTH, THH, TTT, TTH, THT, THH}\}$$

But, we have the two occurrence that are listed twice. These occurrences are, in fact, the intersection of events **A** and **B** for which we solved for earlier. We must take out one of each from our list of occurrences for $A \cup B$:

$$A \cup B = \{\text{HHH, HTH, TTH, THH, TTT, THT}\}$$

We see there are 6 occurrences for this union of events, thus

$$P(A \cup B) = \frac{6}{8} = \frac{3}{4} = 0.75 = 75\%$$

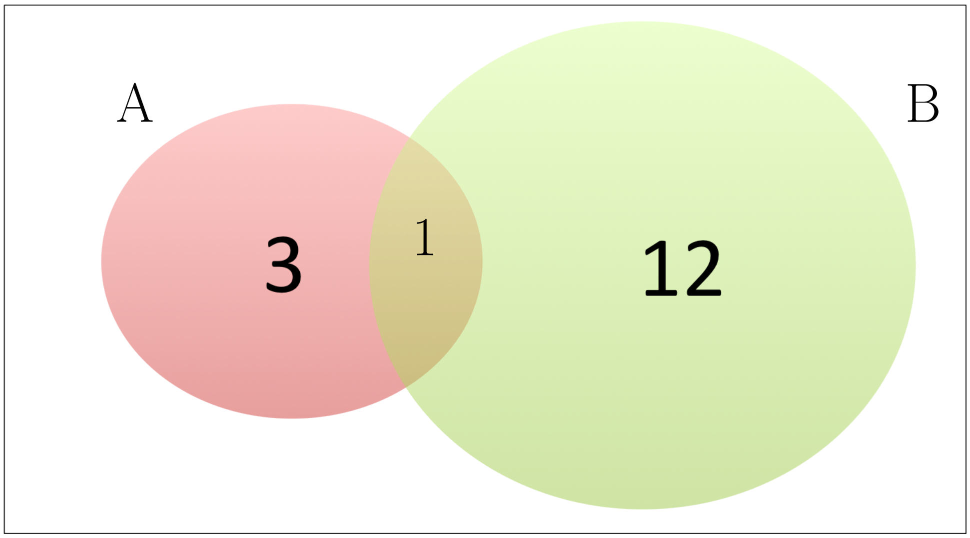
Alternatively,

$$\begin{aligned} P(A \cup B) = P(A) + P(B) - P(A \cap B) &= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} \\ = 1 - \frac{1}{4} = \frac{4}{4} - \frac{1}{4} &= \frac{3}{4} = 0.75 = 75\% \end{aligned}$$

4. Sometimes for two events **A** and **B**, it is impossible for them to occur at the same time. If this is the case for events **A** and **B**, then $P(A \cap B) = 0$. Mathematicians call two events like this "**Mutually Exclusive**".

Example: What is the probability that a random card drawn from a deck of 52 cards is a Jack or a heart?

Firstly, since we are only drawing one card, the size of the sample space is 52. Next, let's define **A** as the event that the card is a Jack, and **B** as the event that the card is a heart. With a Venn Diagram we can get a better visualization of the problem:



The 3 in the red circle represents the Jacks in the deck that are not hearts, and the 12 in the green circle represents the hearts in the deck that are not a Jack. The 1 that is lying in the intersection of these two events represents the Jack of hearts.

Looking at this picture allows us to skip some mathematical computations to find the probability, as we can just add these numbers up to get the number of occurrences of **A** or **B**. After that, we can just divide that number of occurrences by the size of our sample space, which is 52.

$$3 + 1 + 12 = 16$$

$$P(A \cup B) = \frac{16}{52} = \frac{8}{26} = \frac{4}{13}$$

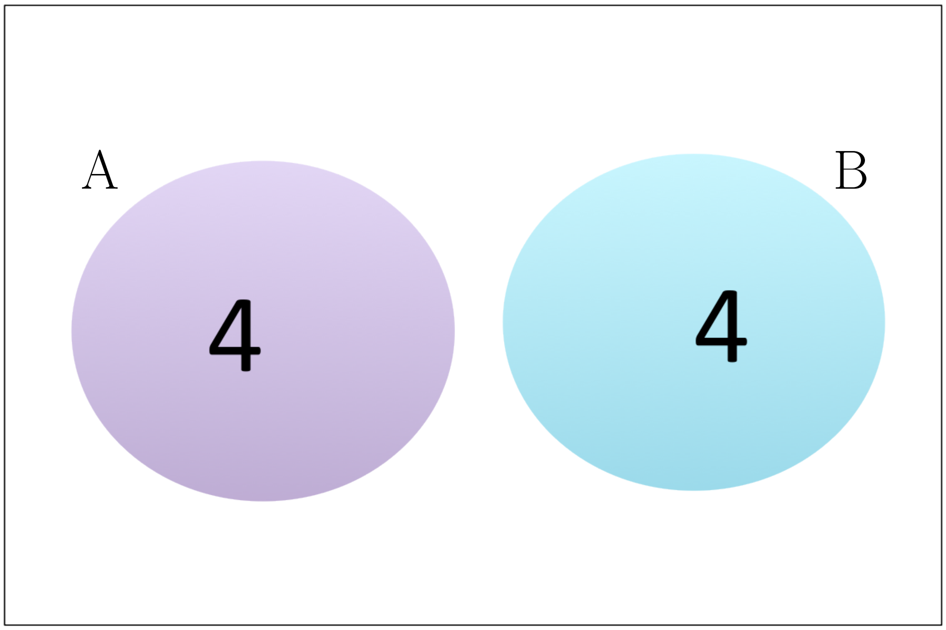
The alternative solution, following from our definition of a union of events, is:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) =$$

$$\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{17}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example: What is the probability that a random card drawn from a deck of 52 cards is a King or an Ace?

Firstly, since we are only drawing one card, the size of the sample space is 52. Next, let's define **A** as the event that the card is a King, and **B** as the event that the card is an Ace. Let's use another Venn Diagram to illustrate the problem:



Because we are looking to draw a King **OR** an Ace, this is a union probability. Also notice that these two events cannot happen at the same time in one card draw, so these events are mutually exclusive. It is easy to see in our picture, as the two circles do not intersect. Therefore:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) =$$

$$\frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{4}{26} = \frac{2}{13}$$

Example: We roll a pair of dice, one after the other. If we are looking at the sum of the dice, what is the probability that we roll an odd number that is greater than 5.

The first thing we always want to do is find the number of possible outcomes in the sample space. Since there are 6 possible outcomes for the first die, and for each of those outcomes we have 6 more when we roll the second die, in our sample space there are:

$$6 * 6 = 36 \text{ possible outcomes.}$$

A good way to visualize why this is true is listing the outcomes in the shape of a square:

$$
\begin{array}{cc}
 & 6 \\
6 & \boxed{\begin{array}{c}
\{1,1\}\{1,2\}\{1,3\}\{1,4\}\{1,5\}\{1,6\} \\
\{2,1\}\{2,2\}\{2,3\}\{2,4\}\{2,5\}\{2,6\} \\
\{3,1\}\{3,2\}\{3,3\}\{3,4\}\{3,5\}\{3,6\} \\
\{4,1\}\{4,2\}\{4,3\}\{4,4\}\{4,5\}\{4,6\} \\
\{5,1\}\{5,2\}\{5,3\}\{5,4\}\{5,5\}\{5,6\} \\
\{6,1\}\{6,2\}\{6,3\}\{6,4\}\{6,5\}\{6,6\}
\end{array}}
\end{array}
$$

Let **A** be the event that we roll an odd number that is greater than 5. When working with dice, it is easier to write down every two-dice occurrence that meets the condition of our event (an odd number greater than 5), rather than use a venn or tree diagram. Thinking about what two-dice occurrences give us a sum that is odd and greater than 5, we only need to consider rolling a 7, 9, or 11.

For 7 we have the following occurrences:

$$\{ \{1,6\}, \{2,5\}, \{3,4\}, \{4,3\}, \{5,2\}, \{6,1\} \}$$

For 9 we have the following occurrences:

$$\{ \{3,6\}, \{4,5\}, \{5,4\}, \{6,3\} \}$$

For 11 we have the following occurrences:

$$\{ \{5,6\}, \{6,5\} \}$$

Counting all these two-dice occurrences that meet the condition of our event **A**, we have the following probability:

$$\mathrm{P(A)} = \frac{12}{36} = \frac{1}{3} = 33\%$$

Problem Set

"*" indicates challenge question

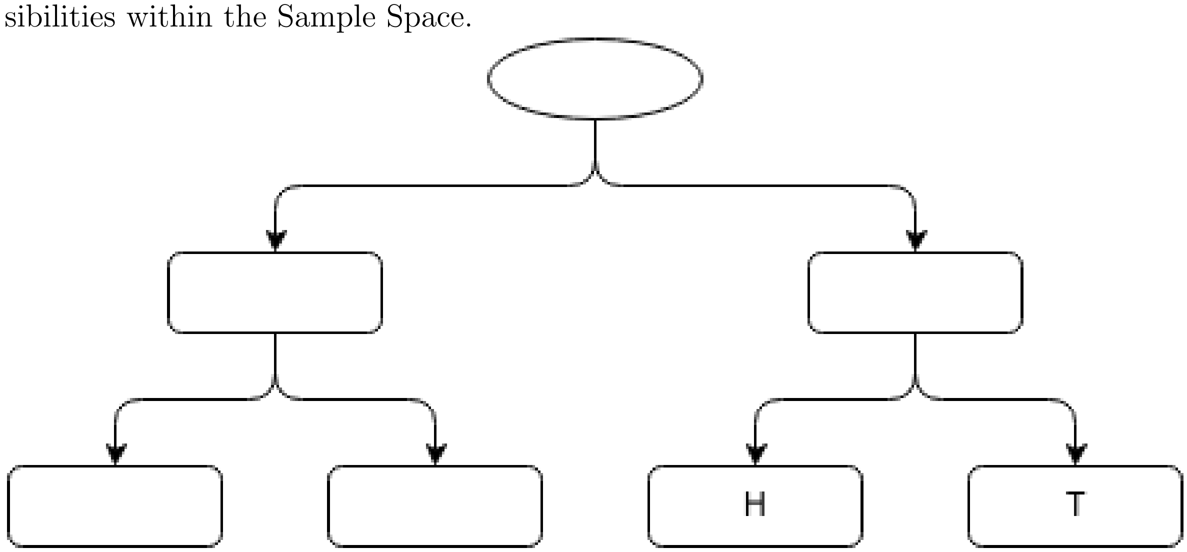
1. In Major League Baseball, one of the most important probabilities measured is whether or not a player will make a hit. This probability is calculated using their batting average, which is collected over the course of each season. What type of probability is this (theoretical or experimental)?

2. One die is rolled three times, with each roll being recorded.

   (a) How many possible outcomes are in this sample space?

   (b) Let **D** be the event that the values of the three rolls are increasing by 1 from least to greatest ({1,2,3}, {2,3,4}, {3,4,5}, {4,5,6}). Let **E** be the event that the values of the three rolls are decreasing by 1 from greatest to least ({6,5,4}, {5,4,3}, {4,3,2}, {3,2,1}). Find $P(D \cup E)$ - the probability of event **D** or **E** occurring.

3. A coin is flipped 3 times. Below is an incomplete tree diagram that represents half of the corresponding Sample Space. Complete the tree diagram then find the total number of possibilities within the Sample Space.



4. Which of the following are not probabilities? Explain.

   (a) 105%

   (b) 1

   (c) 0.65

   (d) -0.45

   (e) 0

5. A pair of dice is rolled and we record the two-dice combination.

   (a) What is the probability that both dice will show a one?

   (b) * Two dice are rolled again. One die shows a one, but the other die rolls under the table and is now out of sight. What is the probability that both dice will show a one, considering you have already gotten the first one?

6. You draw two cards from a standard deck of 52 cards.

   (a) What is the probability you will draw two hearts?

   (b) * What is the probability you will draw two cards of the same suite?

7. We roll a pair of dice. If **A** is the event such that the sum of the dice is even, and **B** is the event such that at the sum of the dice is 6, then find $P(A \cap B)$. (Hint: you only need to find the probability of one of these events occurring)

8. There are 6 red balls, 8 blue balls, and 7 green balls in a box.

   (a) If one ball is randomly drawn from the box, what is the probability that the ball will not be red or blue.

   (b) * Now let's consider if the balls are numbered from 1 to 21 with the first 1 to 6 being red, 7 to 14 being blue, and 15 to 21 being green. If three balls are selected, what is the probability of event **X**, such that the values of the balls are increasing by double with each selection ({1,2,4}, {2,4,8}, {3,6,12}, ...). Once a ball is drawn, it cannot be drawn again.

Conditional Probaility

Conditional Probability is the likelihood of an event **B** occurring, given that event **A** has already happened. This probability is written as:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

9. If P(A) = 10%, P(B) = 45%, and $P(A \cap B) = 5\%$, find $P(A \mid B)$

10. Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

11. You are the teacher of a classroom and you are going over the results of two tests. You have found that 30% of your students passed both tests and 45% of them passed the first test. What percent of students that passed the first test also passed the second one.

12. * There are a total of 500 credit card owners. 300 of these owners are with Visa, 200 of these owners are with Mastercard, and 50 of them are with both.

    (a) Introduce events **A** and **B**, then construct a Venn Diagram to represent this information. Be sure to include the number of owners in each respective circle.

    (b) Given that a random card owner is with Mastercard, what is the probability they are also with Visa.

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13. * You are playing a game of "Blackjack" against a dealer. If two, random cards each are dealt to you and the dealer, in alternating order (you are dealt the first card, they are dealt the second), what is the probability that the sum of your two cards is 21? You are given that:

    - All face cards (King, Queen, Jack) have a value of 10
    - All numbered cards have a value that is the same as their number (5 has a value of 5)
    - The Ace has a value of 11.

14. * **The Monty Hall Problem**

This is a famous math problem that deals with probability. If you would like to face a challenge, google this problem!