MATH CIRCLES MARCH 11 - PROBLEMS

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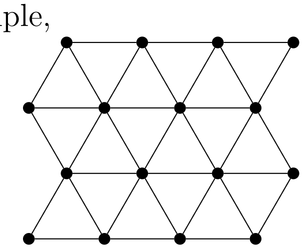
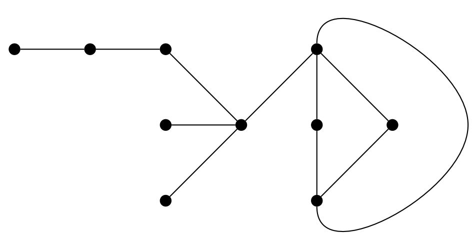
Here are some problems related to game theory. The games aren't necessarily related, and they are in no particular order.

(1) Cops and robbers is a version of "persuit-evasion" games that take place on graphs. The basic idea is that a cop and a robber are placed on vertices of a graph and alternate moving (or choosing not to move) along a single edge to another vertex. The cop wins if the cop and robber end up on the same vertex, and the robber wins if they can avoid capture forever. A graph is called cop-win if the cop can always win, and robber win if the robber can evade capture indefinitely. The cop chooses their starting position first, and the robber chooses second. After that point, they can only move along edges. Which of the following graphs are cop-win, and which are robber-win?

  (a) A path
  (b) A cycle
  (c) A rectangular grid
  (d) A triangular grid. For example,



  (e) A tree
  (f) This graph:



  (g) Any graph that has a cycle of length 4 or larger with no "chords".

(h) See what you can say in general. Don't get down on yourself if you can't find the general strategy. This one is tough.

(2) For the robber win graphs from (1), what happens if there are two cops and they can both move on the cop turn?

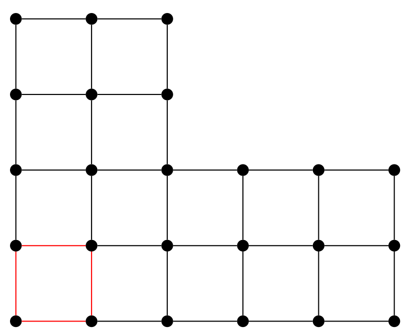
(3) Here are a couple of variations on the subtraction games from the first week:

(a) There is one pile of n stones. The first player may remove as many as they like on their first turn as long as they remove at least one and leave at least one. After that, each player may remove no more than their opponent did on their previous turn.

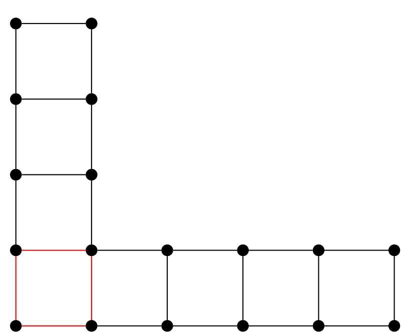
(b) The same rules as the previous game, but each player may remove no more than twice the number of stones their opponent removed on their previous turn.

(c) There are an odd number of stones in a pile. Players alternate removing $1, 2, 3$, or 4 stones. The game ends when all of the stones have been removed and the winner is the player with an even number of stones. Who has a winning strategy?

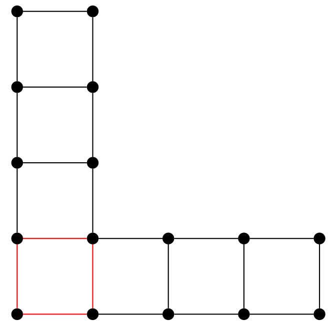
(4) There is a game called chomp. It is played on a partial rectangular grid with the bottom left square identified. Players are allowed to "chomp" a rectangular piece from the top right. The player that is forced to take the bottom left cell loses. For example, maybe the starting grid is



and the first player might (unwisely) chomps the whole inner layer leaving



Then the second player removes a single cell from the bottom row to leave



and it starts to look like Nim. See if you can figure out a strategy for some special cases. It turns out that there is a winning strategy for the player who goes first for any rectangular starting grid, other than 1×1, of course. The strange part is that the proof only asserts existence of a strategy, it doesn't give any hint as to what it is. See if you can figure out why the first player must have a winning strategy.

(5) You are playing a game at a casino and you know that each time you play, you have a 40% chance of winning. You have \$5 and can bet \$1,\$2,\$3,\$4, or \$5 each time you play (you must pick an amount and bet that much ever time). You want to double your money, and will not quit until you have either done it or lost everything. What is the strategy that gives you the best chance of doubling your money? (Try it with less than \$5 first to see what's going on.)

(6) You may have heard this one before. It comes from an old game show called "Let's make a deal". There are three doors and there is a prize behind one of them. There are gag prizes behind the other two. The contestant's objective is to find the prize. They choose one of the doors, and then the host reveals one of the gag prizes to them. They are then allowed to switch to the remaining door or stick with their initial choice. Should the contestant switch or not?

(7) This problem is a little like the previous one, but much more interesting. Cathy and Earl are on a game show. There are three doors which hide a car, a key to the car, and a dummy prize (typically a goat, but I have nothing against goats and therefore don't see why a goat is a bad prize). They are allowed to discuss a strategy before the game starts. In order to win, Cathy must find the key, and Earl must find the car. Cathy goes in first and opens a door. If she finds the key, she leaves immediately and everything is reset (including the key). If not, she can look behind another door. If the key is there, everything is reset and she leaves. If Cathy failed to find the key, they lose. If she found it, Earl gets to play. He gets to open at most two doors and if

he finds the car, they win. They can not discuss anything between their turns. Find a strategy that guarantees a win if Earl gets to play.