MATH CIRCLES MARCH 11 - SOLUTIONS

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(1) It was proved by Nowakowski and Winkler in [4] that a graph is cop win if and only if it is *dismantlable*. See the notes for what this means. In particular, this means that (a), (d), (e) and (f) are cop win, the others are robber win.

(2) Cycles become cop win when there are two cops, and there is no way to know what happens in (g) since the description of the graph is pretty vague. However, rectangular grids also become cop win and it is much more interesting. Here is a description of the strategy:

- The cops place themselves at opposite corners of the grid, say bottom left and top right.
- Once the robber starts moving, the cops close in on the robber by only moving horizontally until one of them is directly above or below the robber after they move. This must happen eventually if the cops move toward each-other. Note that the robber may position himself above or below a cop, or he may move so that he is directly above or below the cop. The point is, at some point the robber will be directly above or below one of the cops and it will be the robber's turn.
- Once this happens, the cop that is directly above or below the robber should match the robber's horizontal moves. That is, if the robber moves right, the cop moves right. If the robber moves left, the cop moves left. This can always be done because the graph is a rectangular grid. If the robber moves up or down, the cop should move vertically toward the robber. Sometimes this will maintain the distance between that cop and the robber, sometimes it will decrease it.
- Meanwhile, the other cop should work to match the robber's vertical position. He can always do this because the robber has to move horizontally "most of the time" because of the other cop. Keep in mind that if the robber moves vertically (up or down) too often, the other cop will catch him. If he keeps moving up, he will eventually be against the top wall and forced to move down on his next vertical move, going toward the cop (if the cop was below). Think about this. Therefore, every time the robber moves horizontally, the second cop

can decrease the vertical difference between him and the robber. Eventually, he will be directly to the right or left of the robber. The second cop should now mimic the robber's vertical moves, and move towards him when he moves horizontally, similar to what the first cop is doing.

- Now there is one cop that is mimicking the robber's horizontal moves and moving toward him when he moves vertically, and the other is mimicking his vertical moves, and moving towards him whenever he moves vertically.
- For argument's sake, imagine that there is a cop directly below and directly to the right of the robber. That is, after the cops move, one is directly below and one is directly right of the robber. If the cop moves down, the first cop moves up, getting closer to the robber, and the second moves down, staying the same distance from him. Similarly if he moves right, one gets closer and the other stays the same distance. If the cop moves up or left, both cops maintain their distance from the robber. However, the robber can not move left and up indefinitely. He must eventually move down or right, making one of the cops closer to him. With this strategy, the cops will eventually catch the robber since the moves where one of the cops gets strictly closer have to happen occasionally. In fact, a modest bound is that one of the cops will get closer at least once every $m + n$ turns if the grid is $m \times n$. Think about this.

(3) (a) For this game, we look to binary expansions again. As long as the binary expansion has at least two ones, it is a next player win. For example, if the pile has 14 stones, it is a next player win since $14 = 1110_2$. In this case, the correct move is to find the right-most one and remove the number of stones corresponding to this one. So if the player is faced with 14 stones, they should remove 2. Notice that the player that makes this move strictly decreases the number of ones in the binary expansion of the size of the pile. Because of the rule that a player may not remove any more stones than their opponent removed on their turn, they can now not decrease the number of ones in the binary expansion. Think about this. It follows that as long as the original pile has at least two ones in its binary expansion, it is a first player win. Otherwise, it is a second player win. In other words, the first player has a winning strategy unless the pile has size a power of 2. This solution was shamelessly stolen from [1]

(b) This game is much more tricky than the previous one. It relies on what is known as Zakendorf's Theorem. See the notes for this Theorem and its proof. The first player has a winning strategy as long as the number of stones in the pile is not a Fibonacci number. However, sometimes the first player will leave their opponent with a Fibonacci number, but they still have a winning strategy. This may seem counterintuitive since leaving your opponent with "good" numbers is a bad idea. However, if you were to draw a game tree for this, there would be more to the states than just the number in the pile. The maximum that can legally be removed must be taken into account. The strategy is as follows:

- If the number of stones in the pile is a Fibonacci number, you are out of luck.
- Otherwise, write the number of stones in its Zakendorf decomposition. That is, if there are k stones, write $k = F_{i_1} + F_{i_2} + \cdots + F_{i_n}$ where the F_{i_j} are Fibonacci number in increasing order, and none of them are consecutive Fibonacci numbers. See the notes for more on this. The correct move is to remove F_{i_1} stones from the pile.

The proof that this strategy works is quite tedious.

(c) We can analyze this game by considering how many stones there are in the pile, and whether the person whose turn it is has an even or odd number of stones accumulated. It turns out that we can just look at the number of stones there are modulo 6. That is, the remainder when the number of stones in the pile is divided by 6. Therefore, we will consider 12 states for this game: $(0, O), (1, O), (2, O), (3, O), (4, O), (5, O), (0, E), (1, E), (2, E), (3, E), (4, E), (5, E)$ where the first coordinate is the number in the pile modulo 6, and the second coordinate is either even or odd. Let's label the states $(0, O), (5, O)$, and $(1, E)$ by P, and the rest by N. In order for this to make sense, we need to show that each N state has a move to a P state, and each P state has only moves to N states. For example, the state $(1, 0)$ has a move to $(0, O)$. The move is to take one stone. To see this, first observe that taking one stone changes the number of stones from $1 \pmod 6$ to $0 \pmod 6$, so the resulting state is either $(0, O)$ or $(0, E)$. The person whose turn it was will now have an odd number $+1$ stones, which is even. This means the person who just moved has an even number of stones, and the number of stones left in the pile is $0 \pmod 6$ which is even. There were an odd number of stones at the beginning, so there are still an odd number of stones between the table and the two players. Therefore, the player whose turn it is next must have an odd number of stones. This means removing one stone sends $(1, O)$ to $(0, O)$, which is a P state. Similar (as well as long and tedious) analysis verifies that every N state has a move to a P state, and no P state has a move to an N state. This means that if you start on an N state, you can force your opponent to always leave you an N state. Eventually you will be in an N state where the number of stones in the pile is less than 6, since the total keeps going down. Consider the state $(4, E)$ where the number of stones left is actually 4, not just 4 (mod 6). It is your move, and you have an even number of stones. You can win from here by taking all the rest. What about the state $(4, O)$ where the number of stones is actually 4? Well, here you take three stones. You will now have an even number, and your opponent is left with one stone that they must remove and end the game, which you will win since you have an even number of stones. Similar analysis can complete the verification that this strategy works. Note that this game and solution was taken from something I read on stack exchange. You can follow the link in [2] to read it for yourself.

(4) This game is pretty neat. A general winning strategy is not known, though you can work it out for some small examples and computers can work it out for larger ones. This problem asks you to prove that there is a winning strategy for the first player in the case where the starting state is a rectangular grid. Here is how it goes: We know that either the first player or the second player can force a win in any combinatorial game, so suppose the second player can force a win. That means that the second player can always leave their opponent with a P state. The first player then removes the top right cell only. The second player then responds by sending the first player to a P state. Since the first player only removed the top right cell, no matter what state the second player left them with (which must be a P state), the first player could have left them with it. Therefore, the first player has a move to a P state, which is a contradiction since we assumed the first player did not have a winning strategy.

This is interesting because it is purely an existence proof. It gives no hint as to what the strategy is.

(5) The best strategy is actually to bet everything right away. This can be calculated explicitly, but the intuition shines through more clearly for a larger amount of money, say \$1000. Imagine betting \$1 every time. Since there is a 40% chance that you win each time, after doing this 100 times, you should have won about 40 times and lost about 60. This means that you will have lost about \$20. So on average, you should

lose $20/100 = 20 each time. Eventually this will run you out of money with high probability. If you just bet everything, there is one chance to win, and it will happen 40% of the time. Otherwise, you will lose. The intuition is that the smaller the bet, the longer you will have to play to double your money. If you play for a long time, the law of large numbers is more likely to take over and set you on the losing side. This idea was taken from an interesting book that can be found in [3].

(6) The contestant should always switch. If they always switch, they will win with probability 2/3. Here's why: Let's assume the contestant chose right in the first place. In that case, if they switch, they will lose. If they chose wrong initially, one of the remaining doors has the prize behind it, and the other has a dummy prize. The host has no choice but to show the dummy prize, so switching at this point will lead to a win. In summary, if they chose wrong initially and switch, they will win. If they chose right initially, switching will cause them to lose. Since there is a 2/3 chance that they chose wrong in the first place, if they switch, they will win 2/3 of the time!

(7) This one is pretty cool. Here is a strategy. Cathy will open door 2 first. If she sees the key, of course she opens no other doors and leaves. If she does not find the key, she chooses which door to look behind next as follows: if she sees the car, she looks behind door 3, and if she sees the goat, she looks behind door 2. There is a 2/3 chance that she finds the key in this way. Assuming she does, knowing what Cathy's plan was, Earl can tell where the car is if he opens door 3 first. If he finds the car, they win. If he sees the key, that means that Cathy didn't see the key when she opened door 2. Furthermore, since Earl is playing, she found the key, which means the thing she found in door 2 told her to check door 3. Because of the way they planned it, this means she saw the car behind door 2, so Earl should open door 2 to find the car. If Earl sees the goat behind door 3, he knows that Cathy did not see the goat when she opened door 2. This means that she did not open door 1. Since she found the key, it had to be behind door 2 or door 3. The goat is behind door 3, so the key must be behind door 2. Therefore, the car is behind door 1.

References

[1] Schwenk, Allen J., Take-away games. , Fibonacci Quart. 8 1970 no. 3, 225234, 241.

[2] http://math.stackexchange.com/questions/731193/how-to-win-this-game

[3] Rosenthal, Jeffrey S., Struck by Lightning: The curious world of probabilities, Harper Collins Publishers Limited.

[4] Nowakowski, Richard; Winkler, Peter, Vertex-to-vertex pursuit in a graph, Discrete Math. 43 (1983), no. 2-3, 235239.