Math Circles
March 25th 2015
Paradoxes
Instructor: Jordan Hamilton
NOTES

1 Introduction To Paradoxes

Let's start with a 'proposition':

Proposition 1. $2 = 1$.

Proof. Take two numbers A and B.

$$\begin{aligned}
A &= B \\
A^2 &= BA \\
A^2 - B^2 &= BA - B^2 \\
(A+B)(A-B) &= B(A-B) \\
A+B &= B \\
B+B &= B \\
2B &= B \\
2 &= 1
\end{aligned}$$

□

Wait, what?! This is an example of a paradox.

We will examine the three main types of paradoxes and then a whole bunch of paradoxes. Along the way we'll see many different areas of mathematics while exercising our brains trying to understand the paradoxes!

1.1 The Three Types of Paradoxes

We are interested in three major types of paradoxes:

1. **Fallacies** (or falsidical paradoxes)

   Definition 1. A **fallacy** is a statement or argument that gives us something contradictory from seemingly correct assumptions.

These usually rely on a hidden or overlooked mistake in the presentation.

2. **Truthful Paradoxes** (or verdigal paradoxes)

   Definition 2. A **truthful paradox** is a statement or argument that seems like it is contradictory, but is actually correct.

   These sometimes result from trying to apply an abstract situation to reality. Another common source is that something might seem really surprising but is actually true.

3. **Antinomies**

   Definition 3. An **antinomy** is a type of statement in which it does not make sense to ask about truth. A paradox arises when we ask about if that statement is true or not.

   Self-referencing statements are a good source of antinomies.

1.2 Our First Three Paradoxes

A Fallacy: We've already seen an example of a fallacy above, in that $1 = 2$ proof.

A Truthful Paradox: Let's examine the *Coin Rotation Paradox*: Take two identical coins and put them side-by-side and touching. Fix one and start rotating the other around it without slipping. After only half a rotation the coin has rotated all the way around! For an animated picture check the following link out: Coin Animation

You can do this in real life, there's no trick to the animation or the description! So what's going on? It turns out that the coin is rotating around two axes at the same time. It's own centre and the centre of the second coin. So rotating it half way around the second coin also causes it to also rotate half way around it's own centre which adds up to one full rotation. Even though this seems strange, we are able to justify it. It does go against how we think it should though! That's why we call it a truthful paradox.

Some Antinomies: There are some common antinomies that we've surely seen at some point before:

This statement is false.

The sentence below is true.
The sentence above is false.

These self-referencing sentences are neither true nor false. We get a paradox by asking if they're true or not.

2 Paradox - Mathematical Induction

2.1 Introduction to Mathematical Induction

Let's start with a brief introduction to mathematical induction.

Suppose we want to prove an infinite number of (related) statements are true, call them P_n for $n = 1, 2, 3, \ldots$. How would we go about doing this in a finite amount of time? We use **Mathematical Induction**.

To prove that P_n is true for all n, we will prove:

1. (Base case) P_1 is true.
2. (Inductive Step) If P_k is true for some k, then P_{k+1} is true.

After we prove both 1 and 2, we can conclude that P_n is true for all n.

Now let's see it in action, on a legitimate example.

Example: For $n = 1, 2, 3, \ldots$ we get

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. First, we prove the base case: $n = 1$. Then the left-hand-side is 1, and the right-hand-side is

$$\frac{1(1+1)}{2} = 1.$$

Therefore the statement is true if $n = 1$.

Now, let's do the inductive step. Suppose that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for some k. We will prove that

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= [1 + 2 + \cdots + k] + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

which is what we wanted to prove. So the statement is true for all n.

□

Now let's see what crazy stuff we can talk about with induction!

2.2 Blue-Eyed Islander Problem

Suppose there are 1000 people living on an island. On that island, 100 of them have blue eyes and 900 have brown eyes. No one on the island ever knows their own eye colour, everyone knows everyone else's eye colour, and no one ever talks about someone else's eye colour. If a person on the island ever discovers their own eye colour, that person must leave the island at dawn the next day. At some point, an outsider comes to the island, calls together all the people on the island, and makes the following public announcement: 'It's nice to see some people with blue eyes like me on this island'. Assuming that everyone on the island is logical, what happens?

Solution:

The answer is that *everyone* leaves the island on the 100th and 101st day! To see this let's start with a simpler problem. Suppose there was just **one** person with blue eyes on the island. Then when the outsider makes the blue eye statement that one person is going to know it's them (as they can't see anyone with blue eyes) and will leave the next day. After that, the rest of the islanders will conclude that they must have brown eyes, or that islander wouldn't have left. So on the second day, the other 999 islanders will leave.

Next, say there are **two** islanders with blue eyes. After the outsider's statement both will think that the other will leave after the first day (as in the case of one islander above). But, after the first day when no one leaves, both blue-eyed islanders will realize that the other MUST see someone with blue eyes, and since no one else has blue eyes they will be able to deduce it's them. Then the two will leave on the second day. The rest will leave on the third day as in the first case.

This extends to any number, so for our 100 blue-eyed islanders after the first 99 days when no one leaves, they will all be able to deduce their eye color and leave on the 100th day. Then the rest leave on the 101st day.

So the solution is pretty crazy! This is a **truthful paradox**, it seems unbelievable but it's not!

3 Paradox - Probability

Some very interesting paradoxes come from probability theory. This is because our brains are:

- Bad at estimating probabilities.
- Easily tricked with probabilities.

Example: What is the probability that you'll flip 5 heads in a row?
Answer: $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$
OK, so if we flip 4 heads in a row, it's pretty unlikely that the fifth flip will be heads right? **Wrong!** There's still a 1 in 2 chance! Doesn't matter what happened before. This is the **gambler's fallacy**.

3.1 The Birthday Paradox

Let's assume there are 365 possible birthdays (sorry, February 29th!). The question is: what is the probability that, in a set of n randomly chosen people, some pair of them will have the same birthday?

Clearly if there are 366 people then the probability is 100%, but how many people do we need to get a probability of 99.9%? Like 360 right? 200? 100?!

The answer is **70** people!

For a 50% probability we only need **23** people! Let's prove this one (proving the statement for 70 follows the same method).

Proof.

- Suppose we have a group of 23 people. Let's figure out the probability that NO two people share a birthday.
- Take persons numbered 1 and 2. The probability that person 2's birthday is different from person 1's is $\frac{364}{365}$.
- Next, the probability that person 3 doesn't share a birthday with person 1 or 2 is $\frac{363}{365}$.
- Keep doing this and we get for person 23, the probability that they don't share a birthday with any of the other 22 people is $\frac{343}{365}$.
- So, the overall probability is: $\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\cdots\left(\frac{343}{365}\right) = 0.492703$.
- That means the probability that some pair of them DO share a birthday is $1 - 0.492703 = 0.507297$ or roughly 50.7%!

$\square$

There was no trickery here, although unbelievable it's a **Truthful paradox**.

4 Paradox - Infinity

Infinity is, perhaps, the most interesting source of paradoxes. Our whole lives, everything we do is finite. The world and everything in it is finite. So when we allow for the infinite, things can get crazy very fast!

Let's start with a source of frequent internet rage: Does $0.9999\ldots = 1$?

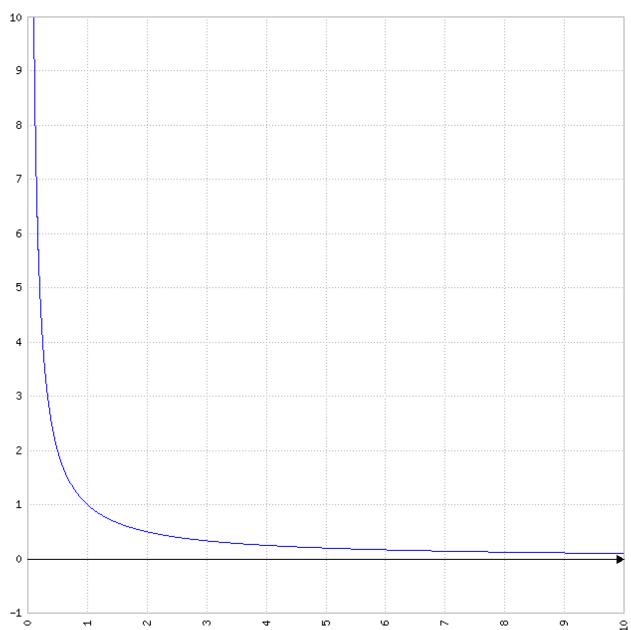
The answer is **yes** but that infinite decimal messes with our brains. Here's the proof:

$$\begin{aligned} x &= 0.9999\ldots \\ 10x &= 9.9999\ldots \\ 10x - x &= 9.9999\ldots - 0.9999\ldots \\ 9x &= 9 \\ x &= 1. \end{aligned}$$

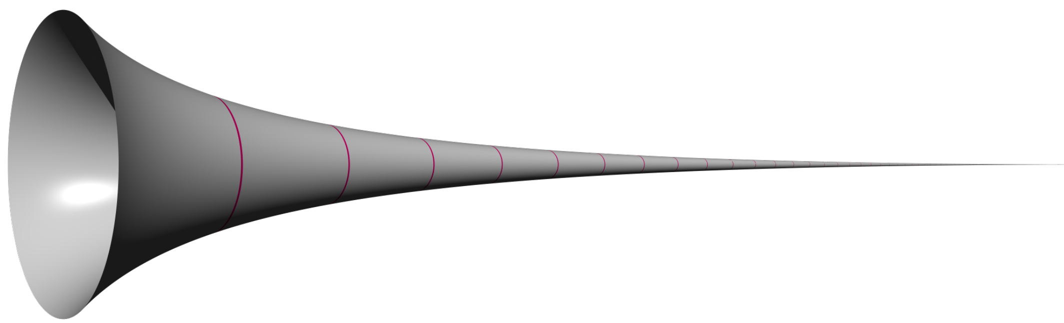
To some this is a **truthful paradox**, to other's it's just a true (and unremarkable) statement (but if you want to cause an argument post this online somewhere).

4.1 Gabriel's Horn

Here's another paradox involving infinities. Consider the function $f(x) = \frac{1}{x}$. It looks like:



If we rotate the graph around the x-axis, from $x = 1$ to $x = \infty$, we get **Gabriel's Horn**:



Using Calculus methods, one can show that this object has *finite* volume and *infinite* surface area! While mathematically sound, our brains are not happy. It seems like we could fill up the horn with a finite amount of paint but we couldn't paint the inside with a finite amount! How can we justify this?

The horn itself is a **truthful paradox**, it does theoretically exist. However, the claim about 'filling/painting the inside' is a **fallacy** since both are actually three-dimensional tasks. No matter how thin our coat of paint, eventually the horn will get so thin that not even one molecule can pass through. We can't make it 'infinitely thin'. It's the two-dimensional surface area that's infinite, not the volume.

5 Some Bonus Paradoxes

Here's a few extra paradoxes just for fun.

5.1 The Grand Hotel Cigar Mystery

This relates to the question about **Hilbert's Grand Hotel** on the exercise sheet.

Suppose cigars are not allowed in the Grand Hotel. But at some point the guest in room 1 decides they really want a cigar. So they go and knock on the door of room 2 and ask the person in there for a cigar. Of course the person in room 2 doesn't have any, but now THEY want one too. So the room 2 guest goes to room 3 and asks for TWO cigars (one for themselves and one for the person in room 1). Room 3 doesn't have any, so the guest in room 3 goes to room 4 and asks for 3 cigars. In general, the guest in room N goes to room $N+1$ to get N cigars, one for themselves and $N-1$ for the guest in the previous room. Does everyone get a cigar to smoke, despite them being not allowed?

The answer is no, of course this process never terminates so everyone will be waiting for their cigars forever. This is a **fallacy**.

5.2 The Interesting Numbers Paradox

Is every positive integer interesting?

- I was born in 1986, so that's interesting.
- It's 2015 now so that's interesting.
- Forty is the first number whose letters are in alphabetical order, pretty interesting!

But is EVERY positive integer interesting? It doesn't seem like we could come up with something interesting for every single one, so let's say no, not every positive integer is interesting. But we can prove that every number is!

Proof. Suppose for a contradiction some numbers are not interesting. Define a subset U of the positive integers that contains all the numbers that are not interesting. We've supposed that U is not empty. That means it's got a smallest number, say s. But s is the smallest number that's not interesting? That's pretty interesting! So U must actually be empty and so every number is interesting. □

What's going on here? It's actually an **antimony**, there's no logical way to split up numbers between those that are interesting and those that are not interesting, since interesting is subjective. So asking about truth here doesn't make sense.