Congruent Number Problem Problem Set 1

(i) Characterization of solutions to Pythagorean Theorem. In this question, we find a parameterization of the solutions to the equation $x^2 + y^2 = z^2$.

1. Show that if a prime p divides two of x, y or z, then it must divide the third one of x, y or z.
2. By the previous part, from now on we can suppose that x, y and z share no common divisors other than ± 1. Show now that exactly one of x or y is even. (Hint: Start by assuming that x and y are both even and showing that x, y, z share a common divisor. Then assume that both x and y are odd, say $x = 2k + 1$ and $y = 2\ell + 1$ for integers k and ℓ and then show that powers of 2 on either side disagree).
3. We'll assume from now on that y is the even number. Isolate for y^2, factor and show that the only common factor of $z + x$ and $z - x$ is 2.
4. Use unique factorization to conclude that $z + x = 2m^2$ and $z - x = 2n^2$. Solve for x, y and z in terms of m and n.
5. Conclude that all possible characterizations are given by $x = d(m^2 - n^2)$, $y = 2dmn$, $z = d(m^2 - n^2)$ for integers d, m, n. Also conclude that m and n have opposite parity (that is, they are not both even or both odd) and that they are coprime (that is, the only common divisor is 1).

(ii) In this exercise, we will show that 1 is not congruent by hand. To do this, we will use the method of infinite descent.

1. First, we will show in parts that $x^4 + y^4 = z^4$ has no integer solution provided $xyz \neq 0$. We lose nothing by showing that there are no positive solutions so we shall do this. First, show that in the equation $x^2 + y^4 = z^4$ as in the previous exercise, we may suppose that x, y and z share no common factor by reducing the equation by the common factor if necessary.
2. Suppose that we have a smallest positive solution (x, y, z) (in the sense of z being minimal). Now using the previous exercise on the equation $x^2 + (y^2)^2 = (z^2)^2$, we may suppose that we are in one of two cases:
   i. Case 1: $x = 2mn$, $y^2 = m^2 - n^2$ and $z^2 = m^2 + n^2$.
   ii. Case 2: $x = m^2 - n^2$, $y^2 = 2mn$ and $z^2 = m^2 + n^2$.

   In each case, show that we can generate a smaller solution to the equation $x^2+y^4 = z^4$. *Hints:* For case 1, consider y^2z^2. For case 2, argue as follows:
   i. Since $y^2 = 2mn$ conclude that $m = 2u^2$ and $n = v^2$ (or $n = 2u^2$ and $m = v^2$; we focus on the first case).
   ii. Since $z^2 = m^2 + n^2$, we can use exercise 1 to find integers r and s such that $z = r^2 + s^2$, $m = 2rs$ and $n = r^2 - s^2$. Show that r and s are squares and that you can find a smaller solution to $x^2 + y^4 = z^4$.
   iii. Thus we decrease forever (infinite descent). We cannot descend forever and so we have reach a contradiction. We conclude that no solution exists.
3. Conclude that 1 is not a congruent number. Hint:

i. Assume that $(a/d)^2+(b/d)^2 = (c/d)^2$ is our rational solution and that $ab = 2d^2$ for positive integers a, b, c, d with d non zero. Any common divisor of a and b divides both c and d. So assume that a and b have no common divisor.

ii. Since $ab = 2d^2$, we may write $a = 2k^2$ and $b = \ell^2$ for some integers k and ℓ. Show that $4k^4 + b^2 = c^2$ and conclude that

$$\frac{c+b}{2} = r^4 \qquad \frac{c-b}{2} = s^4.$$

iii. Solving yields $b = r^4 - s^4$ and from before $b = \ell^2$. The above exercise disproves the existence of such integers.

(iii) Using the fact that 1 is not a congruent number, show that $\sqrt{2}$ is irrational.

(iv) Assume the Birch-Swinnerton Dyer Conjecture. Which of the following squarefree numbers are congruent? Which are not? Which of your proofs did you assume the Birch-Swinnerton Dyer Conjecture? Can you exhibit a triangle in these cases thus removing the need to assume the Birch-Swinnerton Dyer Conjecture?

$$9 \qquad 10 \qquad 13 \qquad 14 \qquad 55 \qquad 107$$

(v)

1. There are many tricks to show that a number is congruent. Here is one way. Suppose a, b, c, d are positive integers. Suppose there are nonzero integers x, y, z, w such that $ax^2 + by^2 = cz^2$ and $ax^2 - by^2 = dw^2$. Show that $abcd$ is a congruent number. (Taken from: "The Queen of Mathematics: An Introduction to Number Theory" p. 47). Hint: Show that

$$(c^2z^4 + d^2w^4)^2 \pm abcd(4xyzw)^2 - 4(cdz^2w^2 \pm 2abx^2y^2)^2 = 0.$$

where the $\pm$ signs on either side are the same. Then consider a triangle with side lengths

$$\frac{U-V}{Y} \qquad \frac{U+V}{Y} \qquad \frac{2X}{Y}$$

where

$$\begin{aligned} X &= c^2z^4 + d^2w^4 \\ Y &= 4xyzw \\ U &= \sqrt{X^2 + abcdY^2} = 2(cdz^2w^2 + 2abx^2y^2) \\ V &= \sqrt{X^2 - abcdY^2} = 2(cdz^2w^2 - 2abx^2y^2) \end{aligned}$$

and write the first hint using these values.

2. Use $(a, b, c, d) = (1, 1, 1, 7)$ and find solutions (x, y, z, w) to the two given equations to show that 7 is a congruent number.
3. Use $(a, b, c, d) = (13, 1, 1, 1)$ and find solutions (x, y, z, w) to the two given equations to show that 13 is a congruent number.
4. (Challenging) Use $(a, b, c, d) = (1, 1, 1, 23)$ and find solutions (x, y, z, w) to the two given equations to show that 23 is a congruent number (the numbers x and y are rather large in this case!).