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# Grade 6 Math Circles

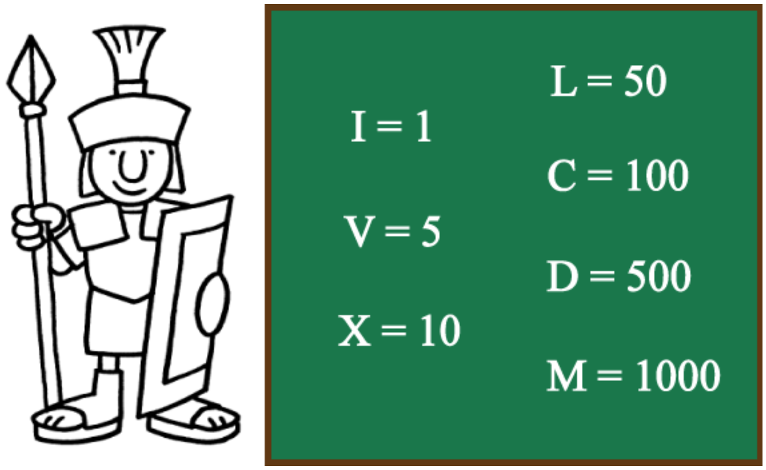
February 9 & 10 2016

## *Ancient Number Systems*

Numbers are vital to many aspects of life. We use them to describe time, keep track of money, do math and science, invent new technologies, predict the future, call grandparents, assign identities and property... you get the idea. Modern life (or any form of organized communal arrangement!) would be impossible without numbers. This is why many different ancient civilizations independently invented numerical systems. Today, we are going to learn how to use some of these ancient number systems.

## Ancient Romans

Roman numerals use figures from the Latin alphabet to represent various values. In modern day, we use different columns to represent numbers. We have the ones column, the tens column, the thousands column, and so on. The tens column is 10 $\times$ bigger than the ones column, which comes before it. The hundreds column is 10 $\times$ bigger than the tens column, which comes before it. Because of this, we say that our number system is "Base 10" or "decimal": we have symbols for the values 0 - 9, but when we want to say "one more than 9", we simply zero out the ones column and add 1 to the tens column to get the number 10. The Romans used a base 10 system as well, but theirs is a little more complicated than ours because in addition to the number 10, the number 5 was also very important to them. They also did not have place values, so the "base" term is a little more general. We say the numbers 10 and 5 are important to them because they have symbols for 1, 10, 100, 1000, and 5, 50, 500. The Roman number system is below.




Symbols are generally placed from left to right in order of greatest value to smallest value. When a symbol is followed by a symbol of equal or smaller value, add these values together. For example, an I after a V or an X means "add one" so VI = 6 and XII = 12. There are some additional rules:

1. An I before a V or an X means "subtract one" so IV = 4. In general, when a symbol is followed by another symbol of larger value, subtract the smaller number from the bigger number.

2. I and V can only be subtracted from symbols with values up to X. For example, you cannot write VL to represent 45. You must write XLV.

3. X and L can only be subtracted from symbols with values up to C.

4. C and D can only be subtracted from symbols with values up to M.

5. Always look for the notation that would result in the smallest number of symbols (ie. write XX and not VVVV).

6. There are never more than 3 of the same symbol in a row. For instance, you would not write XIIII to represent 14. Instead, you would write XIV!

7. Never put more than one smaller number in front of a larger number. For example, you could not write IIX to represent 8. How would you write 8? VIII

8. To signify *even larger* values, we draw a horizontal line over a symbol which means "multiply the value of this symbol by 1000". $\overline{\mathrm{VI}}\mathrm{II} = (6 \times 1000) + 2 = 6002$. Numbers with a line over them go in front of the rest of the symbols.

Note that the Romans did not have a symbol for zero. In fact, having a symbol for zero (or even a concept of it) was a very rare thing to see in the numerical systems of ancient civilizations.

Writing Roman numerals are easiest if you do one place value at a time. If I want to write 645, I write 600 first (DC), then I write 40 (XL), and finally I write 5 (V). So 645 is DCXLV.

*Exercises:*

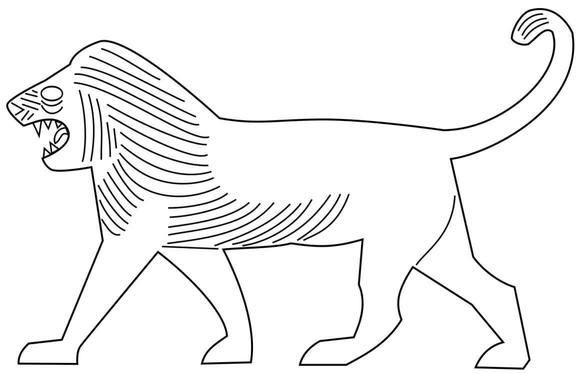
1. Convert the following into modern-day numerals.
   (a) V 5
   (b) IX 9
   (c) XLIII 43
   (d) XXXII 32
   (e) XLVIII 48
   (f) DCCLII 752
   (g) MCCVI 1206
   (h) MMCMXCIII 2993
   (i) $\overline{\text{XLII}}$DC 42,600
   (j) $\overline{\text{LVII}}$CCCLXIV 57,364

2. Translate the numbers into Roman numerals.
   (a) 13 XIII
   (b) 42 XLII
   (c) eighteen XVIII
   (d) 91 XCI
   (e) eighty-two LXXXII
   (f) one-hundred twenty-one CXXI
   (g) 460 CDLX
   (h) 579 DLXXIX
   (i) nine-hundred thirteen CMXIII
   (j) one-thousand two-hundred thirty MCCXXX
   (k) 2592 MMDXCII
   (l) 7436 $\overline{\text{VII}}$CDXXXVI

3. Perform the following operations without converting first. Answer in Roman numerals.
   (a) XIX + XXII = XLI
   (b) XLVII + LXXIII = CXX
   (c) XLV − VIII = XXXVII
   (d) LXXVII − XLVIII = XXIX

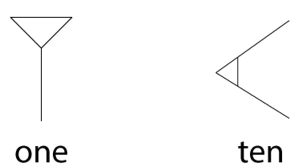
4. Correct the following Roman numerals and give their value in modern-day numerals.
   (a) XLVIIII → XLIX = 49
   (b) CLLVX → CCV = 205

## Babylonians

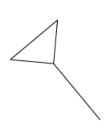


The ancient babylonians were well known for their mathematical prowess. They made many astronomical and geometric calculations that were beyond their time and also invented the abacus. Their choice of number system proved to be very intelligent as well; the babylonians employed a base 60 numerical system. 60 is called a superior highly composite number because it can be divided evenly (with no remainder) by many numbers: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. As a result of this property of 60, the Babylonians were able to split the quantities they were using easily and rarely had to employ the use of fractions. Their influence still exists in math and science today: consider the 360 ($60 \times 6$) degrees in a circle and the use of base 60 for time (60 seconds in a minute, 60 minutes in an hour). These practices survived specifically in these applications because we often need to divide circles and time into smaller units (examples: the use of acute angles, determining how long one quarter of an hour is), and base 60 makes it easier to do that.

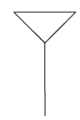
The Babylonians only had two symbols that were used in various combinations to represent 59 non-zero numbers:




Later on, they tilted the one symbol to represent a zero:




These symbols were used very similar to the way the Romans used their symbols to represent different numbers (by adding). One difference is that the Babylonians arranged their numbers in columns, the same way that we do. Since we are a base 10 system, each column is a "power of ten": for us, the column on the right is the ones column, the column to the left of that is the 10s column, the column to the left of *that* is the 100s column, and so on. Each column is ten times as large as the column to the right of it. The Babylonians had a base 60 system, so each column was *sixty* times as large as the column to the right of it. The Babylonians grouped their one and ten symbols together to represent how many units were in each column. For example, if I wanted to represent the number one, I would need to put a one symbol in the ones column like this:



If I wanted to represent the number 61, I would put a one symbol in the 60 column and a one symbol in the ones column like this (notice that a space is left between columns to distinguish between them):

This is exactly the same thing as putting a 1 in the tens column and a 1 in the ones column to represent the number 11 in modern times: it means add 1 ten and 1 one. In the Babylonian case, it means add 1 sixty and 1 one.
The numbers 60 and 70 are shown below:

sixty:

"Add 1 sixty and 1 zero" - note, this use of zero to be a "placeholder" like in the modern day 10 or 101 was a later development. Earlier, the 60 would look just like a 1!

seventy:

"Add 1 sixty and 10 ones"
How would you represent the number 80? (hint: there can be more than one symbol in a column).

Example 1: Write the number ninety-three in modern day script.

I know this seems like an odd example, but it will help you understand the Babylonian use of columns a little bit better. Because ours is a base 10 system, I divide nintey-three by 10 to see how many tens are in this number. Since $93/10 = 9.3$, I get that there are 9 tens in ninety-three so I put a 9 in the tens column:

| x 10 x 10 | x 10 | x 1 |
|---|---|---|
| | 9 | |

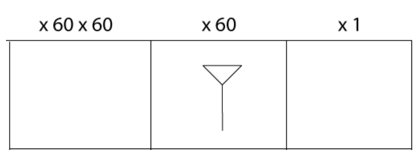
Then, how many ones are left over in this number (what is the remainder)? I know I had 9 tens, so I say that ninety-three minus $9 \times 10 = 3$ so there are 3 ones left over. I put 3 in the ones column:

| x 10 x 10 | x 10 | x 1 |
|---|---|---|
| | 9 | 3 |

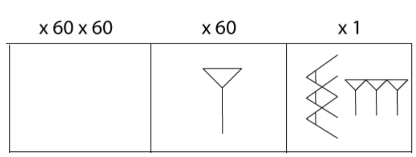
So we can write ninety-three as 93 using modern-day symbols.

Example 2: Write 93 in Babylonian numerals.

Since the Babylonians are a base 60 system, I divide 93 by 60 to see how many sixties are in this number. $93/60 = 1.55$ so there is only 1 sixty in this number. I place a one symbol in the sixties column:



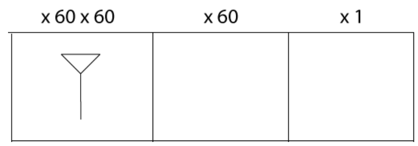

How many ones are left over? $93 - (1 \times 60) = 93 - 60 = 33$ so there are 33 ones left over. Remember, anything less than 60 goes in the ones column, and anything less than $60 \times 60$ goes in the sixties column. We can write the number 33 using 3 ten symbols and 3 one symbols. We place the number 33 in the ones column:



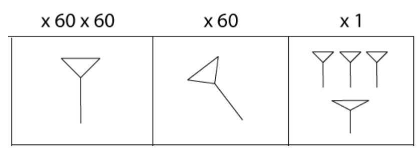

So the symbol for 93 in Babylonian numerals is: ("1 sixty plus 33 ones")

Example 3: Write 3604 in Babylonian numerals.

The number 3604 is greater than (or equal to) $60 \times 60 = 3600$ so this is the first time we are using the column two to the left of the ones column. Instead of dividing by 60 first, we divide by 3600: how many 3600s are in the number 3604? One! So we place a one symbol into the $60 \times 60 = 3600$ column:




How many 60s are left over? $3604 - (1 \times 3600) = 3604 - 3600 = 4$. So there are no sixties left over, but there are 4 ones left over. We zero out the sixties column and put a 4 in the ones column:




So we write 3604 as:

*Exercises*:

1. Covert the following numbers into Babylonian numerals

   (a) 42 

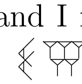
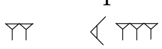
   (b) 84 

   (c) 150 

   (d) 732 

   (e) 6072 

   (f) 12325 

2. I'm building a rectangular hanging garden in my backyard, and I need to know how much space to allocate for it. If I want my garden to be  metres long and  metres wide, how much area should I clear for it? Give your answer in Babylonian numerals.

   Length is 25 m and width is 133 m. Multiply to get area = 3325 $m^2$ or:

    $m^2$

# Some Others...

## Ancient Mayans

The ancient Mayans used a base 20 numerical system that comprised of three symbols: a zero symbol (), a one symbol (•), and a five symbol (—). The numbers are written vertically; that is, higher place values were stacked on top of lower place values. In the modern decimal system, higher place values are placed to the left of smaller place values:

| $10^3 = 10 \times 10 \times 10 = 1000$ | $10^2 = 10 \times 10 = 100$ | $10^1 = 10$ | $10^0 = 1$ |
|---|---|---|---|
| Thousands Place | Hundreds Place | Tens Place | Ones Place |

Since the Mayans used base 20, their place values looked like this:

| | | |
|---|---|---|
| $20^3 = 20 \times 20 \times 20 = 8000$ | 8000s place | • |
| $20^2 = 20 \times 20 = 400$ | 400s place |  |
| $20^1 = 20$ | 20s place |  |
| $20^0 = 1$ | ones place |  |

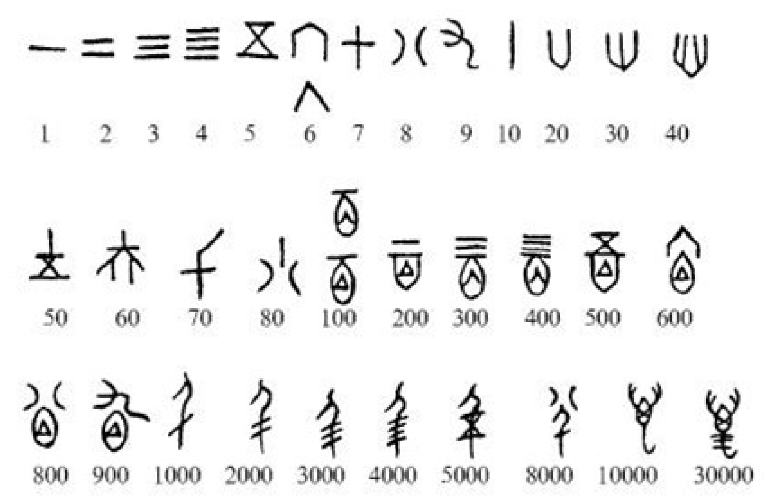
The symbols above represent the number $(8000 \times 1) + (400 \times 0) + (20 \times 10) + (1 \times 7) = 8207$ (Make sure there is enough space between each place value!)

Notice that within each place value, the higher value symbols are written underneath lower value symbols. The  symbol was only needed to "zero out" a place value (like the tens column in 101 to distinguish it from 11). This is similar to the Babylonian system. In fact, the Mayan, Babylonian and modern decimal system are all "positional number systems" because the value of the symbols depend on what place value the symbols are placed in. In the 10's column, a 1 means ten but in the ones column, a 1 is just... 1.

It is thought that base 10 is common in many number systems because humans have 10 fingers. The theory is that the Mayans (who didn't wear shoes) used their toes to count as well – ten fingers and ten toes makes (base) 20!

## Shang Oracle Bone Script

The ancient Chinese during the Shang dynasty over 3500 years ago used tortoise shells ("oracle bones") for divination. On the shells were etched various elaborate symbols that represented words and numbers. Some of the numbers are pictured below:



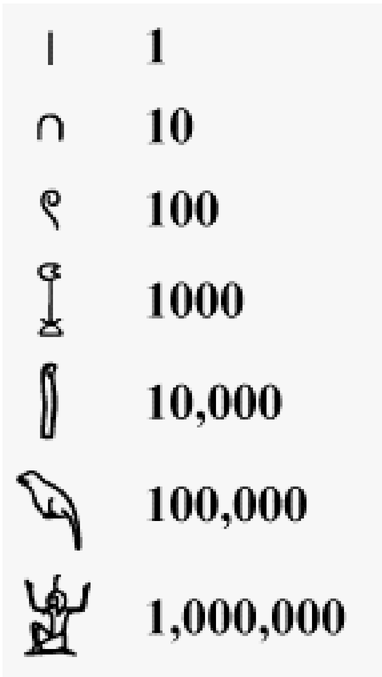

This system is similar to the Roman one. Higher value symbols are placed to the left of lower value symbols and the value of the number is determined by adding symbols together. For example, the symbols  together represent the number $(10,000 + 500 + 30 + 5 = 10,535)$. This system is *not* a positional system. The symbol for 1000 is always a symbol for 1000, no matter where in the number it appears.

The shapes for words and numbers found on oracle bones evolved over time and eventually gave rise to the modern Chinese script.

Notice that there was no symbol for the number zero. The ancient Chinese had no need for a zero because they did not use a positional number system.

## Ancient Egyptians

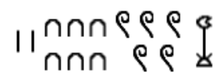
The hieroglyphic script of the ancient Egyptians extended to numbers as well. Their base 10 system is represented by the symbols below.

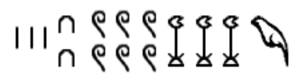



Their numbers were written from right to left; that is, symbols that represented bigger numbers were placed to the right of symbols that represented smaller numbers. Symbols placed together were then added to give the value of the number they represented.

Note: the ancient Egyptians didn't have a symbol for zero.

*Exercise*: Try writing 1562 and 103,623 in hieroglyphics!

1562 is 

103,623 is 

## Ancient Greeks

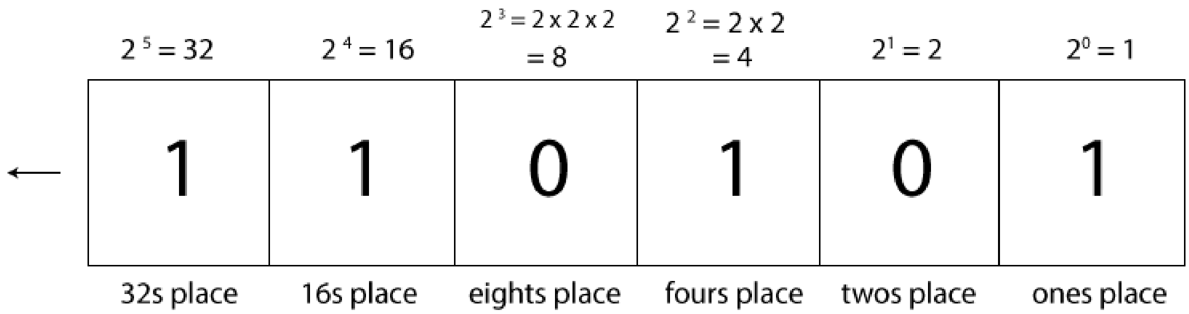
The ancient Greeks just used their regular alphabet to represent numbers as well:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $\alpha$ | alpha | 1 | $\iota$ | iota | 10 | $\rho$ | rho | 100 |
| $\beta$ | beta | 2 | $\kappa$ | kappa | 20 | $\sigma$ | sigma | 200 |
| $\gamma$ | gamma | 3 | $\lambda$ | lambda | 30 | $\tau$ | tau | 300 |
| $\delta$ | delta | 4 | $\mu$ | mu | 40 | $\upsilon$ | upsilon | 400 |
| $\epsilon$ | epsilon | 5 | $\nu$ | nu | 50 | $\phi$ | phi | 500 |
| ς | stigma | 6 | $\xi$ | xi | 60 | $\chi$ | chi | 600 |
| $\zeta$ | zeta | 7 | o | omicron | 70 | $\psi$ | psi | 700 |
| $\eta$ | eta | 8 | $\pi$ | pi | 80 | $\omega$ | omega | 800 |
| $\theta$ | theta | 9 | ϟ | koppa | 90 | ϡ | sampi | 900 |

Symbols with larger value were placed to the left of symbols with lower value (the opposite of the Egyptians). To figure out the value of the number, just add all the individual symbol values together. Do not use more than one symbol from each table. Writing larger numbers is more complicated so we'll skip it for now. This means we can only represent numbers up to 999 with this system.

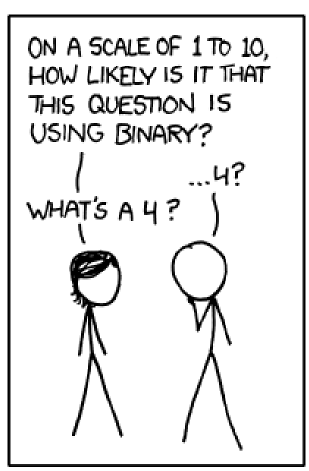
## Binary

Okay, this is kind of cheating a little bit because as far as I can tell no ancient civilization actually used base 2, but binary is so important to the modern day that I am going to mention it anyway. In a base 2 number system, there are only two symbols: a 1 and a 0. The place value is determined the same way the decimal system's is, but of course with a base of 2 instead of a base of 10:




So the symbol 110101 in binary represents the number $(1 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 32 + 16 + 4 + 1 = 53$. Binary is used *everywhere* today in every single electronic device. This is because at its most basic level, a computer only has two possible states: on (1) or off (0). Using these two states, we can do so many incredible things – create a network that connects everyone in the world with each other, send probes to ***Pluto***, calculate trillions of digits of pi, and even this!

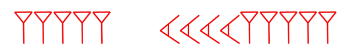
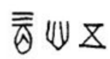
Remember that whenever you use a positional system like binary, Mayan, Babylonian, or modern decimal that you can extend the place values as far as you want. This is what the arrow above represents. I could have included the $2^6$ column, the $2^7$ column, and so on.




https://xkcd.com/953/

# Activities

1. Write the numbers 345, 765, 1400*, 234, 8321*, and 102,233* using each *ancient* numerical system above (not binary). Do not convert the marked numbers to Greek numerals. One of the solutions is provided below.

| Number | Roman | Babylonian | Mayan |
|---|---|---|---|
| 345 | CCCXLV |  |  |
| | Shang | Egyptian | Greek |
| |  |  | τμε |

2. Perform the following operations without converting.

   (a) 

   (b) 

   (c) 

   (d) 

3. Find the numerical value of each Greek word (accents redacted for simplicity).

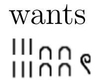
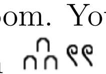
   (a) δρακων (dragon) = 975

   (b) εδρα (hydra) = 110

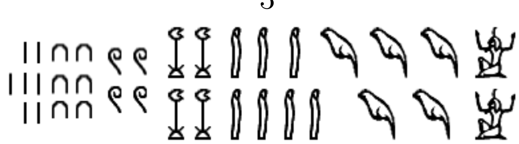
   (c) τιταν (Titan) = 661

   (d) δημοκρατια (democracy) = 554

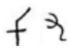
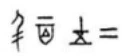
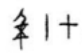
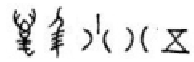
   Please note that no one in ancient Greece would actually assign a numerical value to words this way. Numbers and letters had the same symbols but were used differently.

4. You have been tasked with designing a great pyramid for your pharaoh Khufu. This magnificent tomb will be his final resting place, so he wants to make sure that he's got lots of room. You are told to build a pyramid that is  m tall with a square base of length  m. How much space will ~~your~~ Khufu's great pyramid hold? Answer in Egyptian numerals.

   Reminder: $V = l \times w \times \dfrac{h}{3} = 230 \times 230 \times (146/3) = 2,574,467 \text{ m}^3$

   Or: 

5. The Stupendous Seer of Shang China (you) is approached by a lesser Diminutive Diviner who asks you for help seeing the future. He has written down everything that he's prophesied while in a future-seeing trance on his oracle bone but can no longer decipher what much of it says. So far, he has decrypted:

   "Supper is in  days,  hours,  minutes, and  seconds" but needs help with the rest. What does the oracle bone say?

   The first number is $70 + 9 = 79$. The second is $2000 + 200 + 50 + 2 = 2252$. The third is $5000 + 10 + 7 = 5017$, and the last number is $30,000 + 4000 + 80 + 8 + 5 = 34,093$.

6. Coming into the modern day, it appears that the positional number system has won out, but how come? What are the benefits to using a positional number system?

   It allows us to write very large numbers without having to create new symbols.

7. Write a different number using each of the number systems above (one for each system). Give them to someone else to translate and then translate their numbers.

8. Create your own number system. Choose a base, whether it will be positional or not, if it will have a zero, what your symbols will be, etc.

9. A simple Caesarian Shift Cipher works like this: each letter is given a value from 1 to 26 (or 0 to 25) like below:

| A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

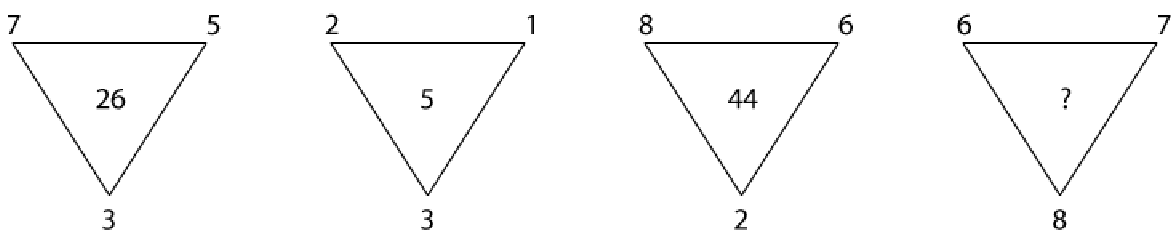
| S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|
| 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |

The person encrypting the message decides on a number called the "key". He then *shifts* the entire table above by that value. For example, if I chose a right shift of 3, then the letter A would now be represented by the number 4. If I chose a left shift of 3, the letter A would be represented by the number 24. I then use this new table to write a message. The message I create can use the shifted letters in the table or the shifted numbers. For example, if I want to decrypt the word "cat" with a right shift of three, I might write down the letters "FDW" or the numbers "6/4/23". I send this message to the decryptor, who is given the original table that is above and the key (3). Suppose I send him the message "6/4/23". Since he knows I shifted the table by 3 to get those numbers, he will know that the message means either "IGZ" or "CAT". He picks the one that makes sense.

On the next page is a Caesarian cipher for you to solve. The key is the answer to the riddle below.

What is the missing number?

Multiply the two largest together. Subtract the smallest number multiplied by itself.




The missing number is $(8 \times 7) - (6 \times 6) = 20$

Solve the Caesarian cipher.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| XXV | XLII | XXV | XXXVIII | XLV | XXXV | XXXIV | XXV |
| 25 – 20 | 42 – 20 | 25 – 20 | 38 – 20 | 45 – 20 | 35 – 20 | 34 – 20 | 25 – 20 |
| E | V | E | R | Y | O | N | E |
| XXXIX | XXXV | XXXV | XXXIV | XXXV | XXXVIII | XXXII | XXI |
| 39 – 20 | 35 – 20 | 35 – 20 | 34 – 20 | 35 – 20 | 38 – 20 | 32 – 20 | 21 – 20 |
| S | O | O | N | O | R | L | A |
| XL | XXV | XXIII | XXXV | XXXIII | XXV | XXXIX | XXXVIII |
| 40 – 20 | 25 – 20 | 23 – 20 | 35 – 20 | 33 – 20 | 25 – 20 | 39 – 20 | 38 – 20 |
| T | E | C | O | M | E | S | R |
| XXXV | XLI | XXXIV | XXIV | XXII | XLV | XXXVIII | XXXV |
| 35 – 20 | 41 – 20 | 34 – 20 | 24 – 20 | 22 – 20 | 45 – 20 | 38 – 20 | 35 – 20 |
| O | U | N | D | B | Y | R | O |
| XXXIII | XXV | | | | | | |
| 33 – 20 | 25 – 20 | | | | | | |
| M | E | | | | | | |

Now write your own cipher. This time, use binary. See who can solve it!