Math Circles - Surfaces

Night 2

Tyrone Ghaswala - ty.ghaswala@gmail.com

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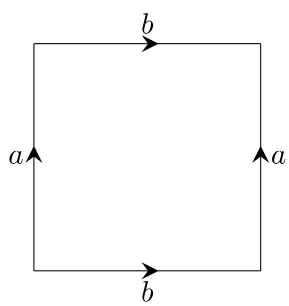
Last time we talked about 2-dimensional surfaces, and introduced the Euler characteristic and planar models.

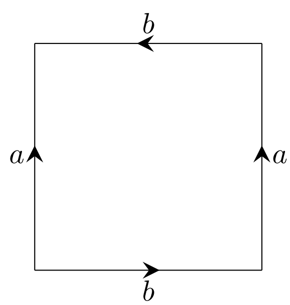
Planar Models

So, now that we have the Euler characteristic, in theory we should be able to start proving some things about surfaces. However, in practice this is not turning out to be the case. The current way of calculating the Euler characteristic of a surface seems prohibitively tedious and difficult, so we need some help. Enter planar models!

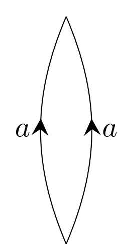
This is a way of visualising surfaces without having to imagine 4 dimensions. We will see it is a useful tool for just about everything! Let's have a look at the planar models of some familiar surfaces (see if you can work these out for yourself, some of them appear in the exercises).



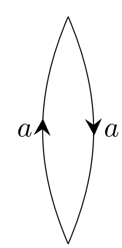
Torus



Klein Bottle



Sphere



Projective Plane

Word Representations

Drawing these planar models out every time does help for cutting and pasting, but it does sometimes get tedious. Instead we can simply encapsulate all the information of a planar model from a word representation. So, how do you make a word representation you ask? As follows!

1. Draw your surface as a single polygon.

2. Start at any vertex, move in any direction (clockwise orcounter-clockwise).

3. Read off the letters and every time you have an arrow going in the opposite direction you are, put an inverse sign.

So, we have some word representations as follows:

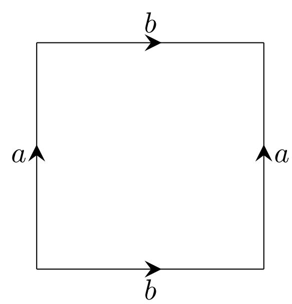
$$\begin{aligned}\text{Torus: } & aba^{-1}b^{-1}\\ \text{Klein bottle: } & abab^{-1}\\ \text{Projective Plane: } & aa\\ \text{Sphere: } & aa^{-1}\end{aligned}$$

At this point you might be wondering, why do we even care about all of this. Sure, it makes it easier to visualise, but it doesn't seem to be getting us any closer to proving anything. Or maybe it does! Here are two reasons why we care about planar models and their word representations.

1. It makes calculating the Euler characteristic easier, and
2. For the first time, we actually have a chance of proving that two surfaces are the same!

Computing the Euler characteristic

Let's look at the first advantage with the Torus. From the exercises, we know in order to turn the Torus into a polyhedron, and then count the numbers of vertices, edges and faces without making a mistake, we would probably have to be some sort of Greek god. However, with a planar model it becomes much easier. Let's look at it again.



At first glance, if we count the number of vertices on this picture we get 4. However, this is incorrect because when we glue all the arrows back together, we find that some of these vertices are actually glued together and thus are the same. Let's count these slowly.

The vertex at the top left is at the 'end' of the a edge, thus it is glued to the top right vertex, so those two are the same. The top right vertex is also the 'end' of the b edge, so it is the same as the bottom right vertex. The bottom right vertex is also the''start' of the a edge, so it is the same as the bottom left vertex. Thus, all the vertices are the same and there is only 1 vertex.

If we count the edges, we see that the two a edges are actually the same (after we glue it back up) and the two bs are also the same. Thus there are 2 edges.

Of course, there is only one face, the square itself. So, with this in mind we have

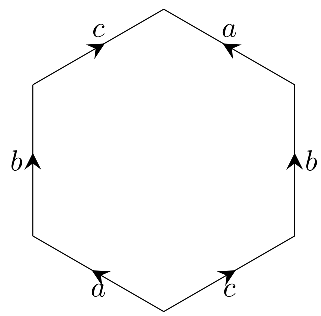
$$\chi(T) = V - E + F = 1 - 2 + 1 = 0.$$

Much easier!

Proving the equivalence of two surfaces

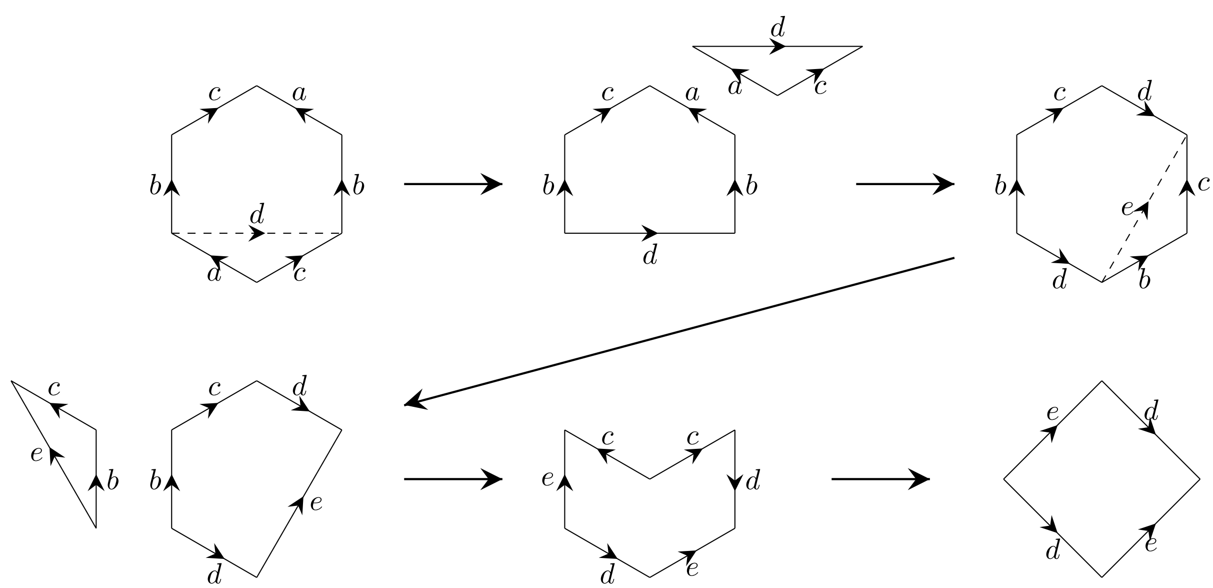
So far, we've only had ways of proving that two surfaces are *not* equivalent. Let's now do an example to see how we can use a technique called cutting and pasting (very high-tech stuff!) to actually show two surfaces which appear to not be equivalent are actually equal.

Here's a question, what is the surface with word representation $abca^{-1}b^{-1}c^{-1}$? Is it something we've seen before or can we prove it's something new? Let's start by looking at it's planar model to figure out it's Euler characteristic. If it doesn't agree with the Euler characteristic of anything we've already seen, we can definitely say it's not a surface we've seen before. It's planar model is:



This has 2 vertices (top right, top left and bottom are all the same, as are the other three), 3 edges and 1 face, so its Euler characteristic is 0. Unfortunately, we can't conclude anything from this, but we can make a guess that maybe it's the Klein bottle or the torus (it can't be anything else we've seen because nothing else has Euler characteristic 0).

Well, let's start playing around to see if it is. The general strategy is as follows: make cuts between vertices, keeping track of which edges are to be glued back together, and glue other vertices back together. This isn't a very clear description, but it will become clear with this example. We will perform the following cuts and pastes on our planar model, and see what happens! The dotted lines are where we are going to cut.



Now this last shape looks extremely familiar! It's a torus and what we have shown here is that

$$abca^{-1}b^{-1}c^{-1} \sim ede^{-1}d^{-1} \sim \text{ a torus!}$$

Amazing! It's a fun exercise to try and imagine how gluing the edges from the first hexagon actually gives you a torus (we've shown it indeed does!).

Orientability

So we've played around with cutting and pasting a little, but we're still not satisfied. This is because we know

$$\chi(T) = \chi(KB) = 0.$$

This doesn't tell us that these are different, but if we take a planar model for the Torus and try and cut and paste it into a planar model for a Klein bottle, we really struggle to do so. Remember, just because we haven't been able to doesn't mean it's impossible, but our guts tell us it is. We need something else besides the Euler characteristic. It's extremely frustrating that we can't seem to tell the torus and Klein bottle apart.

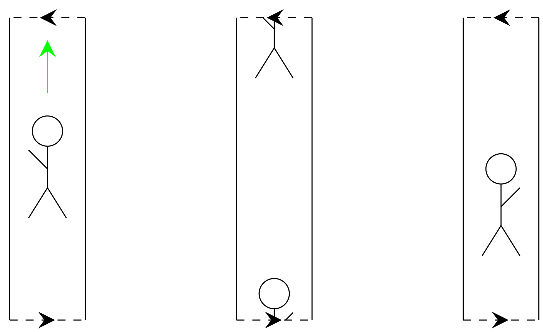
Enter, orientability.

Notice that when we tried cutting and pasting the planar model of the torus, at every step if we look at the word representation, we always have letters matching up with opposite orientations. For example, the two word representations we have seen for the torus are

$$aba^{-1}b^{-1} \quad \text{and} \quad abca^{-1}b^{-1}c^{-1}$$

and in both cases, we have exactly one of each pairs of letters going forward, and the other backward. However, this isn't the case with the standard word representation of the Klein bottle, $abab^{-1}$. So, what's going on here? It turns out the answer lies in looking at the Möbius strip more carefully.

Picture a little 2-dimensional man with one arm (his right one) living in a Möbius strip (here he actually is in the Möbius strip, not above or below it, but in it, much like we are inside our 3-dimensional universe). If he walks (as best as a 2-dimensional man can manage) around the Möbius strip once, when he gets back to the same spot, his right hand will be missing and he will now have a left one! Somehow the notion of right handed and left handed does not make sense when we're dealing with a Möbius strip, or when we're dealing with any surface with a copy of the Möbius strip inside it.

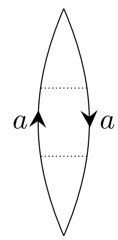


Being left or right handed doesn't make sense on a Möbius strip

Let's define this formally.

Definition. A surface is **non-orientable** if it contains a copy of the Möbius strip. It is **orientable** otherwise.

So, is the projective plane orientable? Well, we built it out of a Möbius strip so we know there's a copy of it living somewhere in the surface. So the answer is "no". However, more enlightening is looking at the planar model. We see that between the dotted lines below, lives a Möbius strip!



A Möbius strip in a projective plane

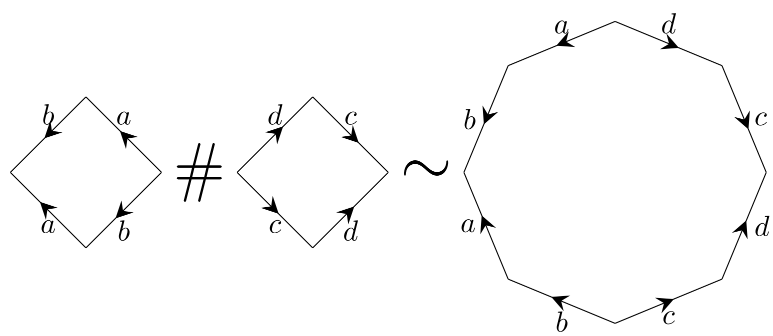
In fact, if you think hard enough and do the exercises, we can see that if a word representation of a surface has a pair of letters, both going forwards or backwards, then the surface is not orientable! For example, the surface $aabcdd^{-1}cb^{-1}$, whatever it is, is not orientable because it has a pair of cs with the same direction (it also has the pair of as but we only need one pair for it to be non-orientable).

Connected sum and the Euler characteristic

One of the important exercises was asking about planar models for the connected sum, and how the Euler characteristic changes when we take the connected sum of two surfaces.

Well, after a bit of work in the exercises you can convince yourself that if we have two word representations W and X for two surfaces A and B, then a word representation for $A\#B$ is WX. You simply squish the two words together! This can be thought of in terms of the planar models as follows: Take the vertex from A that you start the word from, the vertex in B that you start the word from, glue the two models together at the point, and stretch the surface to separate the point and make a single polygon.

For example, if we take two tori, say given by $aba^{-1}b^{-1}$ and $cdc^{-1}d^{-1}$, then a word representation for $T\#T$ (which we're pretty sure is the torus with two holes) is $aba^{-1}b^{-1}cdc^{-1}d^{-1}$! To see this with planar models we have:



So, with this in mind, we can now generalise this idea and (by one of the exercises) show the following theorem.

Theorem 1. *If A and B are any two surfaces, then $\chi(A\#B) = \chi(A) + \chi(B) - 2$.*

Pretty neat hey? It finally feels like we're starting to make some progress.