# Intermediate Math Circles
Wednesday November 9 2016
Mathematical Games III

1. Find a winning strategy for the following game: Two players alternate removing stones from a pile, and the player who removes the last stone wins (as usual). The only restriction is that players may only remove a power of 2 on a move (that is, 1, 2, 4, 8, etc.). Find all starting sizes for which the second player has a winning strategy. How does the strategy change if you are only allowed to remove powers of 3? What about powers of 5?

2. The game in this problem is the same as the previous one except players are only allowed to remove perfect squares, not powers of 2. That is, the legal numbers of stones to remove are $1, 4, 9, 16$, etc. Show that there is no largest pile size for which the second player has a winning strategy. [This one is quite tricky.]

3. Find a winning strategy for a single pile subtraction game (like (1) and (2)) where the players are allowed to remove 1, 2, or 6 stones. Choose your own set of allowed numbers and find a winning strategy.

4. Call a number **evil** if there are exactly two 1s in its binary expansion. For example, 3, 5, and 20 are evil since their binary expansions are 11, 101, and 10100, which each have exactly two 1s. Find the sum of all evil numbers which are less than 1024. [Hint: First, try to find the sum of all evil numbers less than 8, less than 16, and less than 32 to try to find a pattern.]

5. Determine whether or not the following are cop win or robber win.
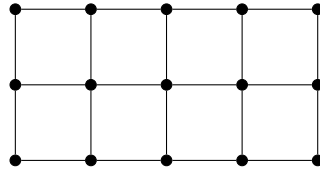
    (a) A path. That is,
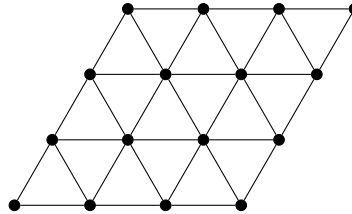

    (b) A cycle. That is,
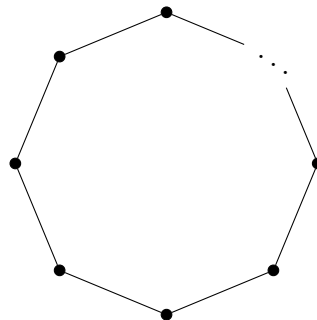
(c) A rectangular grid. For example, a 3×5 rectangular grid looks like

You might think this is a 2×4 grid, which is fine.

(d) A triangular grid. For example,

6. A 6 sided die is rolled twice. What is the probability that the product of the two rolls is

(a) equal to 5,

(b) equal to 6,

(c) equal to 7,

(d) more than 6,

(e) more than 20.

7. A 6-sided die is rolled three times. What is the probability that the sum of the three rolls is more than

(a) 3,

(b) 4,

(c) 15,

(d) 17,

(e) 9.

8. The digits 1,2,3,4, and 5 are arranged to form a 5-digit number. What is the probability that the number is

(a) even,

(b) a multiple of 5,

(c) a multiple of 11.

(d) Answer (a), (b), and (c) with the digits 1 through 6, rather than 1 through 5.

9. Player A and Player B are playing a game where they are flipping a coin repeatedly. The game stops when one of these four things happens:

   (i) A head is flipped first, in which case Player A wins.

   (ii) Two heads are flipped in a row, in which case Player A wins.

   (iii) Two tails are flipped in a row, in which case Player B wins.

   (iv) After the coin has been flipped 5 times, if none of (i)-(iii) has occurred, the game ends and it is a draw.

   This game is entirely based on chance, so there is no winning strategy possible. Here are some questions about the game.

   (a) What is the probability that the game ends in a draw?

   (b) What is the probability that Player A wins?

   (c) What is the probability that Player B wins?

   (d) Answer (a) and (b) when the game is changed so that the "5" in (iv) is changed to "100". You answer can be a (long) sum of numbers. [Hint: This one is also hard.]