



Grade 6 Math Circles

March 28/29, 2017

Probability

Counting Review

Last week we looked at **counting**. Counting is very important when studying probability so let's start off with reviewing some key concepts.

Fundamental Counting Principle (FCP)

If you have to make Choice A **AND** Choice B and there are m options for Choice A and n options for Choice B, then the total number of different ways you can make Choice A and Choice B is $m \times n$.

Factorial

When there is no repetition and you are ordering n objects:

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

Permutations

When there is no repetition, order matters, and you want to permute (ie. order) k objects out of a total n objects:

$${}_nP_k = \frac{n!}{(n - k)!}$$

Combinations

When there is no repetition, order does not matter, and you want to choose k objects out of a total n objects:

$${}_nC_k = \frac{n!}{k!(n - k)!}$$

Introduction to Probability

What is probability? You probably think of it as the “chance” of something happening. Probability is used everyday by people from statisticians to everyone in this classroom! Before you agree to a race or to playing a game, you will think of the probability that you win. Today, we will be learning about the formal definition of theoretical probability.

Theoretical Probability

The probability of event A happening can be calculated with this formula:

$$P(A) = \frac{\text{Number of ways A can occur}}{\text{Total number of possible outcomes (in sample space)}}$$

When calculating these probabilities, knowing how to **count** is very important!

Sample Space

What is a sample space? The sample space of an activity is the set of all possible outcomes of the specific activity. If you are rolling a die, “the sun exploding” is not a valid outcome in the sample space because (though it could happen) it has nothing to do with the die roll.

Exercise: Find the total number of possible outcomes in the sample spaces of each activity. Assume that the order matters when an action is repeated.

1. Flip a coin

2. Flip a coin twice

3. Draw 4 cards from a shuffled deck, putting them back after each draw

4. Draw 4 cards from a shuffled deck **without** putting them back after each draw

Now that we know how to count sample spaces, we can compute probabilities!

Exercise: Find the probability of the following events as fractions and percentages.

1. Flip a coin and get heads
 2. Flip a coin twice and get 1 heads and 1 tails
 3. Draw 4 cards from a shuffled deck, putting them back after each draw, and get all hearts
 4. Draw 4 cards from a shuffled deck **without** putting them back after each draw, and get all hearts

Intersections

The probabilities we have looked at so far have been fairly simple and well suited to turn into math problems. But probabilities are more useful when we can find them for more complicated events. We will now look at how to calculate probabilities for **multiple events**.

Independent

Event A and event B are independent if the outcome of one does not affect the outcome of another. In today's Math Circles, we will only look at independent events.

Intersections

An intersection of events A and B means events A “**AND**” B. In mathematical notation, we write “ $(A \cap B)$.”

Product Rule (AND)

The product rule is used to find the probability of an intersection of independent events:

$$P(A \text{ AND } B) = P(A \cap B) = P(A) \times P(B)$$

Exercise:

Harry and Ron are playing the card game Exploding Snap. During each person's turn, the player takes a card which has a $\frac{1}{3}$ chance of exploding independent of the other cards in the deck. Once a card explodes, the game ends and that player loses. What is the probability that, starting at Harry, the loser is Ron on the second turn of the game?

Exercise: Let's try this problem again using the Product Rule.

Find the probability of drawing 4 cards from a shuffled deck, putting them back after each draw, and getting all hearts.

Union

Mutually Exclusive

If event A AND event B can never occur at the same time, then they are called mutually exclusive. Mathematically, we would say $P(A \cap B) = 0$.

Exercise: Are these events mutually exclusive?

1. A Hogwarts student being sorted into Gryffindor house or sorted into Slytherin house
2. Remus likes to eat chocolate or ice cream

Unions

A union of events A and B means A “OR” B. In mathematical notation, we write “ $(A \cup B)$.”

Special Sum Rule (OR)

The sum rule calculates the probability of a union of **mutually exclusive** events:

$$P(A \text{ OR } B) = P(A \cup B) = P(A) + P(B)$$

General Sum Rule (OR)

The sum rule calculates the probability of a union of **not mutually exclusive** events:

$$P(A \text{ OR } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Exercise:

What is the probability that tossing a 6-sided die results in an even number.

Exercise:

What is the probability that a card randomly drawn from a deck is a jack OR a clubs card?
(Note: drawing a jack and clubs card are independent events.)

Exercise: Weather Report

Suppose each day's weather is independent and each day can only be either sunny (S) or rainy (R). Each day has a $\frac{5}{7}$ chance of sun and a $\frac{2}{7}$ chance of rain. Madame Hooch is throwing a 2-day outdoor sports event on Monday and Tuesday. If one of the days is not clear, that day's events can be moved to Wednesday.

What is the probability that the 2 days of events can be completed due to good weather?

Bonus: Markov Chains

Markov Chains are a type of process where, each **step**, you can move between different **states** in a repeated activity. States are similar to the different outcomes in a sample space.

A **one step probability matrix** is a number grid that tells you the probabilities of going between states in one step. You can find the probabilities of moving from State 1 to State 2 by starting at the State 1 row (on the right) and then the State 2 column (at the top).

You can use the one step probability matrix and the product rule to find the probability for a chain of events in a Markov Chain!

Exercise: Roll up the Rim!

In Tim Horton's Roll up the Rim event, when you buy a hot chocolate you can either Win another hot chocolate or Lose. If you win, then you have a chance of playing again and winning more hot chocolates!

Here is the one step probability matrix of having a winning streak of this event:

$$\begin{array}{cc} & \text{Win} \quad \text{Lose} \\ \text{Win} & \left(\begin{array}{cc} \frac{1}{6} & \frac{5}{6} \\ 0 & 1 \end{array} \right) \\ \text{Lose} & \end{array}$$

Notice that if you start from the state "Lose", there is a 0% probability of winning and 100% probability of losing next.

If you have just won, then what is the probability that:

1. you lose with the next hot chocolate?
2. you win with the next hot chocolate?
3. you win three more times exactly (without buying a new hot chocolate)?

Problem Set

1. Harry and Hermione have these potions in front of them: 1 moves them forward, 2 do nothing, 1 moves them backwards, and 3 are poisons. They choose a potion to drink. What is the probability that:
 - (a) They move forward
 - (b) Nothing happens
 - (c) They are not poisoned
2. What is the size of the sample space for the following events?
 - (a) Guessing a 10-digit pin code
 - (b) Guessing a 10-digit pin code with no repeated digits
 - (c) Picking a winner and first runner up in a competition with 8 participants
 - (d) Picking 2 owls out of the 9 in the owlery to deliver a package
3. Kingsley must visit 7 different cities on one trip.
 - (a) How many different routes can he take?
 - (b) What is the probability the route he takes puts the cities in alphabetical order?
4. The Ministry of Magic is holding a lottery and has sold 2000 tickets. If Draco Malfoy has a $\frac{1}{16}$ chance of winning, then how many tickets did he buy?
5. Mr. Filch is inspecting students' school bags for banned prank items. Each bag independently has a 40% chance of containing a banned item. Mr. Filch selects two students at random to inspect, one at a time. What is the probability that:
 - (a) Both bags have banned items?
 - (b) Both backpacks do not have banned items?
 - (c) Exactly one backpack has a banned item?
6. Albus, Scorpius, and Rose are each trying to brew a potion. Albus' potion has a 90% probability of being effective, Scorpius' has a 98% probability of being effective and Rose's has a 86% probability of being effective. What is the probability that:
 - (a) All three of their potions are effective?

- (b) No one's potion is effective?
- (c) Only Scorpius' potion is effective?
- (d) Exactly one person's potion is effective?

7. Are the following outcomes of events mutually exclusive?

- (a) The number that comes up when you roll a die
- (b) Drawing a face card or a heart card
- (c) Being 11 or 5 years old
- (d) Playing basketball or playing soccer

8. If you roll two dice (assume that the order of the dice matters), what is the probability that the numbers you get in the two rolls add up to:

- (a) 1?
- (b) 12?
- (c) 8?
- (d) Greater than 2?

9. If a number were randomly picked from the whole numbers 1 to 20. What is the probability that it is an odd number or a prime number?

10. Of 50 seventh year Hogwarts students, 13 students are taking Transfiguration, 19 are taking Charms, and 11 are taking both. What is the probability that a student takes:

- (a) Only Charms?
- (b) Both Transfiguration and Charms?
- (c) At least one of Transfiguration or Charms?
- (d) Exactly one of Transfiguration or Charms?

11. There are four different small model dragons in a bag and the four Triwizard champions must each pick one from the bag without looking. Once the dragon is picked, it does not go back in the bag. What is the probability that Harry chooses the Hungarian Horntail if he goes:

- (a) First?

- (b) Second?
- (c) Third?
- (d) Last?

12. **Challenge problem.** For a special promotion, Tim Hortons is selling cups of hot chocolate that give you the chance of winning a small (S), medium (M), or large (L) hot chocolate or win a donut (D). Once you win the donut, you cannot win again (unless you buy a new hot chocolate) but if you win a hot chocolate, there is a chance you will have a winning streak!

Here is the one step probability matrix of having a winning streak of this event:

$$\begin{matrix} & \begin{matrix} S & M & L & D \end{matrix} \\ \begin{matrix} S \\ M \\ L \\ D \end{matrix} & \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

- (a) What is the probability that, if you have just bought a small hot chocolate, the next thing you will win is:
 - i. a small hot chocolate?
 - ii. a medium hot chocolate?
 - iii. a large hot chocolate?
 - iv. a donut?
- (b) If you have just bought a medium, what is the most and least likely thing you'll win next?
- (c) What is the probability that you will win a small, medium, large, and then donut (in that order) if you start with a:
 - i. a small?
 - ii. a medium?
 - iii. a large?
 - iv. a donut?

13. **Challenge problem.** On a game show the contestant is shown 3 doors: A, B, and C. Behind one of the doors is a car and behind the other two are goat. You, the contestant, want to pick the door with the car behind it.

- (a) What is the probabilities that the car will be behind door:
 - i. A?
 - ii. B?
 - iii. C?
 - iv. Either B or C?
 - (b) You decide to pick door A. Before you lock in your decision, the show's host opens up door B and reveals that there's a goat behind it! Now, what is the probability that the car will be behind door:
 - i. A?
 - ii. B?
 - iii. C?
 - (c) Should you switch your pick?
14. **Challenge problem.** There are 35 students in your Math Circles class. Show that the probability that at least two students in the class have the same birthday is 80%.
- Hint:* Find the probability that no pair of students in the class has the same birthday.