Senior Math Circles
Wednesday, March 1, 2017
Problem Set 2

1. Find examples of:
   a) A plane that satisfies $A2$ but not $A1$.
   b) A plane that satisfies $A1$ and $A3$ but not $A2$.
   c) A plane that satisfies A_1 and $A2$ but not $A3$.
2. Show that if a line in an affine plane intersects one of two parallel lines, then it intersects the other.
3. Determine if the following are affine planes:
   a) The plane $\mathbb{R}^3$ where the points are the usual points (points of the form (a, b, c) with $a, b, c \in \mathbb{R}$) and the lines are the usual lines.
   b) The plane $\mathbb{R}^3$ where the points are the usual points (points of the form (a, b, c) with $a, b, c \in \mathbb{R}$) and the lines are planes.
   c) Consider a Euclidean sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Note that every plane of Euclidean 3-space intersects S either in a point, or a circle, or is disjoint from S; and every circle on S arises in this way, i.e. as a plane intersection. Now take points to be just the ordinary points of S, and lines to be circles in S.
   d) Consider S as before but now fix a point on S which we call the north pole. Take as points, the ordinary points of S except for the north pole; and take lines to be the ordinary circles of S passing through the north pole.
4. Proof the following theorem:

   Theorem: Let $(\mathcal{P}, \mathcal{L})$ be an affine plane of order n. Then,
   (i) $\mathcal{P}$ has exactly n^2 points.
   (ii) Each point lies on $n + 1$ lines.
   (iii) Each pencil contains n lines.
   (iv) The total number of lines is $n(n + 1)$.
   (v) There are $n + 1$ pencils of parallel lines.
5. A quadrangle is a set of four points, no three of which are collinear. According to axiom A3, the affine plane of order 3 should have at least one quadrangle.
   a) Give an example of a quadrangle in that plane. (This shows that A satisfies A3.)
   b) How many quadrangles does this plane have? Explain your answer.
6. Draw the affine plane of order 4. Clearly show the points in each line and make a distinction on the elements in each pencil.