Senior Math Circles
Wednesday, March 1, 2017
Solutions - Problem Set 2

1. Find examples of:

   a) A plane that satisfies $A2$ but not $A1$.

   b) A plane that satisfies $A1$ and $A3$ but not $A2$.

   c) A plane that satisfies A_1 and $A2$ but not $A3$.

   Solution:

   a) Consider $\mathcal{P} = \{1, 2, 3\}$ and $\mathcal{L} = \{\{1, 2\}, \{1, 2, 3\}, \{3\}\}$. Notice that for every point that doesn't belong to a line, there is a parallel line that contains the point. However $A1$ is not satisfied since the points 1 and 2 belong to two different lines.

   b) The Fano plane.

   c) Let $\mathcal{P} = \{P, Q\}$ and $\mathcal{L} = \{l_{PQ}\}$. This plane satisfy $A1$. It also satisfies $A2$ vacuously and since there are just two points, it does not satisfy $A3$.

2. Show that if a line in an affine plane intersects one of two parallel lines, then it intersects the other.

   Solution: Let l and m be two parallel lines and k be a line that intersects l. If k does not intersect m i.e. if $k \parallel m$, choose $P = l \cup k$ be the point of intersection of l and k. Then, $P \notin m$ and l and k are two distinct lines through P that are parallel to m. This is contradicting the condition $A2$.

3. Determine if the following are affine planes:

   a) The plane $\mathbb{R}^3$ where the points are the usual points (points of the form (a, b, c) with $a, b, c \in \mathbb{R}$) and the lines are the usual lines.

   b) The plane $\mathbb{R}^3$ where the points are the usual points (points of the form (a, b, c) with $a, b, c \in \mathbb{R}$) and the lines are planes.

   c) Consider a Euclidean sphere $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Note that every plane of Euclidean 3-space intersects S either in a point, or a circle, or is disjoint from S; and every circle on S arises in this way, i.e. as a plane intersection. Now take points to be just the ordinary points of S, and lines to be circles in S.

   d) Consider S as before but now fix a point on S which we call the north pole. Take as points, the ordinary points of S except for the north pole; and take lines to be the ordinary circles of S passing through the north pole.

Solution:

a) This is not an affine plane. It does not satisfy $A2$. Given a line and a point not in that line, there are infinitely many parallel lines that contain the point (lines that don't intersect the given line).

b) This is not an affine plane. For example $A1$ doesn't hold. Given two points, there are infinitely many planes that contain these two points.

c) This is not an affine plane. Choose any pair of points in the sphere. We can find infinitely many lines passing through these two points.

d) This is in fact isomorphic to the real affine plane. This is most easily seen by stereographic projection, as follows. Denote by N the north pole of S. Consider the plane π tangent to S at the south pole, so that S sits on π. For each point (i.e. point of P other than N), the ray connecting P and N intersects π at a point P' of π. So we see that lines on S correspond to lines in the plane π.

4. Proof the following theorem:

Theorem: Let $(\mathcal{P}, \mathcal{L})$ be an affine plane of order n. Then,

(i) $\mathcal{P}$ has exactly n^2 points.
(ii) Each point lies on $n+1$ lines.
(iii) Each pencil contains n lines.
(iv) The total number of lines is $n(n+1)$.
(v) There are $n+1$ pencils of parallel lines.

Solution:

(i) We know that any line contains exactly n points. Let l be any line in $\mathcal{L}$. Let $P_1, P_2, \cdots, P_n$ be the points on the line l. Now, y $A3$, there is some point $B \notin l$. Let $l_1 = l_{P_1B}$ and by $A1$ and $A2$, consider the line l_i to be the line that contains P_i and that is parallel to l_1 for $i \in \{2, 3, \cdots, n\}$. Since all the lines l_i are different and belong to the same pencil, we see that there are n points in each line and we have n lines, thus the plain contains n^2 points. Now Notice that if we choose any point in $\mathcal{P}$, this point is in some line l_i, by the previous exercise. Then we see that the plain contains exactly n^2 points.

(ii) Suppose B is any point on the plane. Choose a line l not containing B. So there are n lines through B and any point in l (Since l has n points). Also there is a unique line through B that is parallel to l. Thus in total we have $n+1$ lines through B.

(iii) Choose any line l and a point $P \in l$. If β is a pencil and l is not in β, then l intersects any line in β at some point in l. Since l has n points we see that there are n lines in β.

(iv) Choose a point P in the plane. We know that there are $n+1$ lines passing through P. Each of these lines belongs to some pencil and we see that each pencil contains n lines. Thus we have in total $n(n+1)$ lines.

(v) Since each of the $n(n+1)$ lines belongs to a unique pencil and each pencil has n lines, we see that there are $n+1$ pencils.

5. A quadrangle is a set of four points, no three of which are collinear. According to axiom A3, the affine plane of order 3 should have at least one quadrangle.

   a) Give an example of a quadrangle in that plane. (This shows that A satisfies A3.)

   b) How many quadrangles does this plane have? Explain your answer.

Solution:

a) Consider the points $1, 2, 4, 5$. No three of them are cointained in a line, so this is a quadrangle.

b) Let's count all the possible subsets of four elements and then we can subtract the number of nonquadrangles as follows:

   Total number of subsets with 4 elements is $C(9,4) = 126$ a combination of 9 (the total number of points) by 4 (number of elements in the subsets). Now we count all the possible sets of 4 elements that don't form a quadrangle, i.e. sets of four elements where three of them are collinear. So, we can choose the line that contains the three points over all the 12 possible lines and once we choose a line we choose a fourth point that is not in the line, which can be chosen in six different ways. Then the number of sets that are not quadrangles is $6(12) = 72$. Then the number of quadrangles is $126 - 72 = 54$.

6. Draw the affine plane of order 4. Clearly show the points in each line and make a distinction on the elements in each pencil.

   Solution: The following picture shows the plane splted into the 5 pencils H, V, P_1, P_2, P_3.