An Introduction to Graph Theory

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What is a graph?

Definition

A graph G is:

- a set $V(G)$ of objects called **vertices**

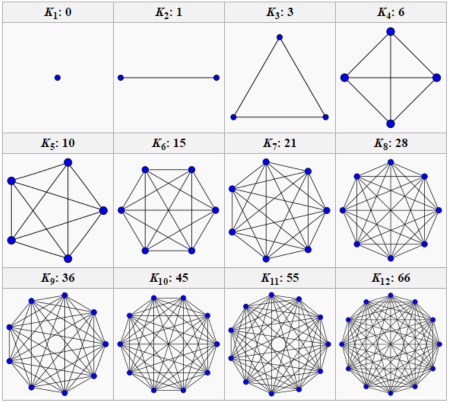
together with:

- a set $E(G)$, of what we call called **edges**. An edge is an unordered pair of vertices.

We call two vertices **adjacent** if they are connected by an edge.

Why do we study graphs?

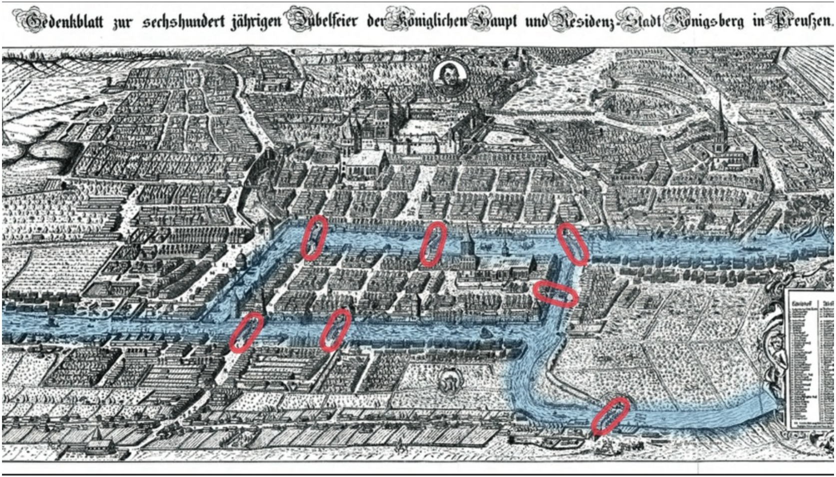
- They have a nice pictorial representation
- Can model many real-world problems using graphs
- Like many topics in discrete maths, they're just nice to think about in their own right.



A few definitions before we get started

- Walk: A **walk** of length k on a graph G is a sequence $v_0, e_1, v_1, e_2, v_2, e_3, ..., e_k, v_k$ where the $e_i, k \geq i \geq 1$ are edges in $E(G)$ and the $v_i, k \geq i \geq 0$ are vertices in $V(G)$.
- Path: A **path** of length K on a graph G is a walk with no repeated edges or vertices.
- Circuit: A **circuit** is a walk where $v_0 = v_k$.
- Cycle: A **cycle** is a circuit with no repeated vertices *except* $v_0 = v_k$.
- Degree: The **degree** of a vertex v is the number of edges touching v.

Königsberg Bridges Problem



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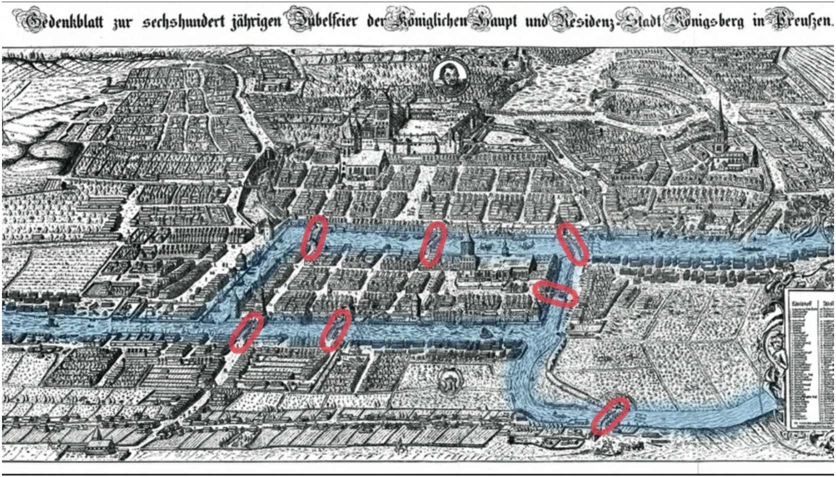

How can we represent this problem on a graph?

Königsberg Bridges Problem

The problem becomes finding a circuit[1] that goes through each edge exactly once. We call this sort of circuit an eulerian circuit.

Eulerian graph: We call a graph **eulerian** if it has an eulerian circuit.

[1] Recall: a **circuit** on a graph G is an alternating sequence $v_0, e_1, v_1, ..., e_k, v_k$ of vertices and edges where $v_0 = v_k$.

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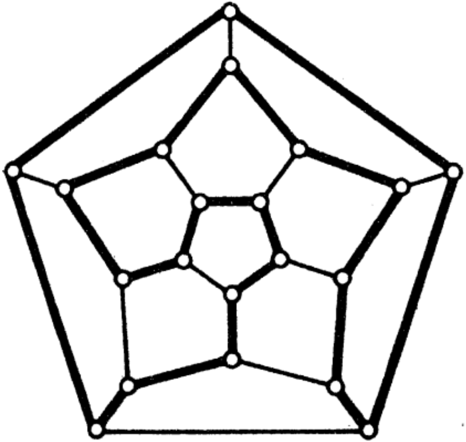
Theorem

A graph is eulerian iff every vertex has even degree.

[1]Recall: a **circuit** on a graph G is an alternating sequence $v_0, e_1, v_1, \ldots, e_k, v_k$ of vertices and edges where $v_0 = v_k$.

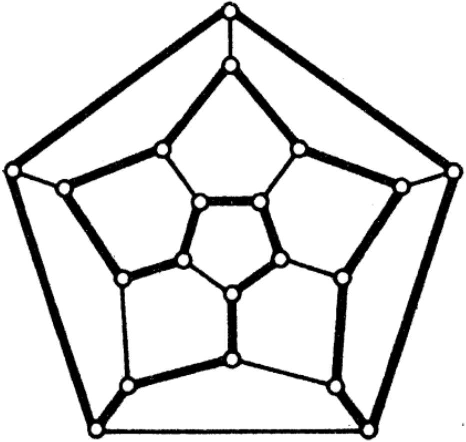
What about visiting every vertex only once?

Hamiltonian graph: A graph is called **Hamiltonian** if it contains a cycle that visits every vertex exactly once.



Named after William Rowan Hamilton, who, in a letter to a friend and in 1856, described a game on the graph below:

- one person marks out five consecutive vertices
- the next person has to use these five consecutive vertices to create a cycle that reaches every vertex.



How can we characterize hamiltonian graphs?

Short answer? We can't! At least, not completely. However, we know some partial results.

Sufficient condition (Dirac, 1952)

Let G be a (simple) graph with at least three vertices. G is hamiltonian if the minimum degree is at least $\frac{|V(G)|}{2}$.

Now, let $\omega(G)$ be the number of components[2] of a graph H.

Necessary Condition

If G is hamiltonian, for every nonempty proper subset of $V(G)$, $\omega(G - S) \leq |S|$.

[1] Recall, two vertices are in the same **component** if and only if you can get from one to the other via a path.

Handshaking Lemma

Theorem

For any graph G,

$$2|E(G)| = \sum_{v \in V(G)} deg(v).$$

We will prove this via a proof technique known as a double-counting argument: we will show that two expressions are equal by demonstrating that they are just **two ways of counting the same thing**.

Handshaking Lemma

Why is it called the handshaking lemma?

- Suppose we have a group of people. We'll represent each person by a vertex, and when two people shake hands, we will draw an edge between them.
- The number of edges is then equal to the number of handshakes.
- If we ask each person v how many times she shook hands (this corresponds to $deg(v)$) and sum up everyone's answers, we will be counting each handshake twice: once for each person who participated in the handshake.

Handshaking Lemma

Corollary

Every graph has an even number of odd-degree vertices.

Proof.

From the handshaking lemma, we have that for any graph G, $2|E(G)| = \sum_{v \in V(G)} deg(v)$. Let's look at this more closely:

$$2|E(G)| = \sum_{v \in V(G)} deg(v)$$

$$2|E(G)| = \sum_{v \in V(G), deg(v) \text{ even}} deg(v) + \sum_{v \in V(G), deg(v) \text{ odd}} deg(v)$$

$$2|E(G)| = 2t + deg(v_1) + deg(v_2) + \cdots + deg(v_k)$$

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Now, note since the left-hand side of the equation is even, the right hand side must be too. Since each of the $deg(v_i)$s are odd, there must be an even number of them. $\square$

How else can we characterize graphs?

We've seen two ways of characterizing graphs:

- Whether or not they are Hamiltonian
- Whether or not they are Eulerian

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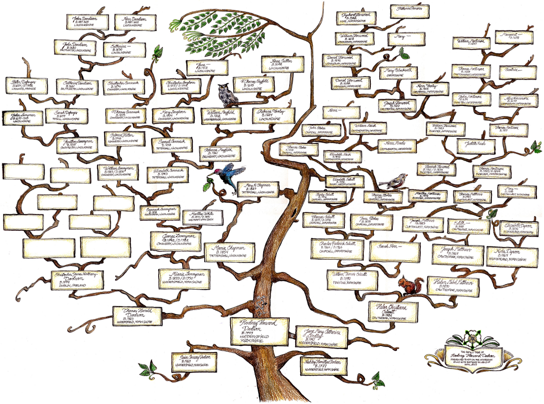
How else can graphs be characterized?

Eulerian circuits and hamilton cycles are both sub-structures we can find in our graphs. A natural question is then: what other structures can we use to characterize graphs?

Trees

A **tree** is a connected[3] graph that contains no cycles.



[1]Recall: a graph is connected if you can get from each vertex to each other vertex via a path.

Theorem

If G is a tree, for any two vertices u and v in $V(G)$, there is **exactly one** path from u to v.

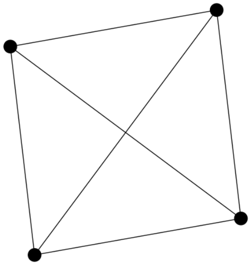
Proof.

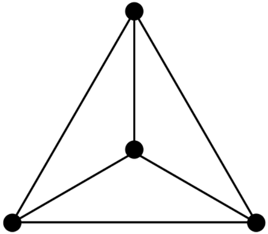
By contradiction. Suppose there are two distinct paths P_1 and P_2 with $P_1 = u, v_1, v_2, ..., v_k, v$ and $P_2 = u, w_1, w_2, ..., w_t, v$. Let x be the first vertex in P_1 that is also in P_2[4]. Let $x = w_a$ for the appropriate a. Then $C = u, v_1, v_2, \ldots, x, w_a - 1, w_a - 2, \ldots, w_1, u$ is a cycle, and so G is not a tree. □

[4] Can we guarantee x exists?

How else can we characterize graphs?

- As we've seen, one of the reasons graphs are fun objects to study is because they have very nice visual representations. We might try to characterize graphs, then, based on what their representations look like.
- Generally speaking, we consider graphs to be equivalent if they have the same structure, *no matter what configuration we've drawn them in*.





Planar graphs

We call a graph **planar** if it can be drawn in such a way that no edges cross.

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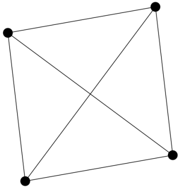
e.g. This graph is planar:



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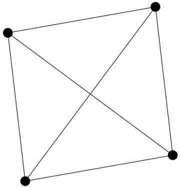
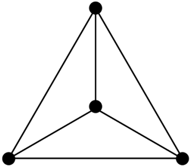


Here is a planar embedding of it:

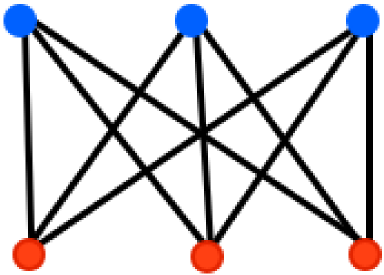


Planar graphs

There is a classical mathematical problem that asks the following: suppose we have three houses, and three resources. Can you hook up each house to each resource by drawing a line from house to resource in such a way that none of the lines cross?

Planar graphs

We can represent this problem as a graph (surprise!...).

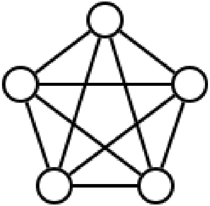


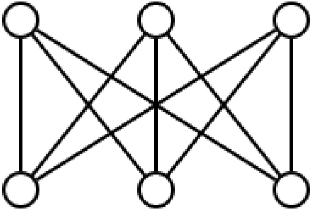
This graph is called $K_{3,3}$ and it is one of the most well-known non-planar graphs. In fact, we can characterize all non-planar graphs using $K_{3,3}$ and one other graph, called K_5.

Planar graphs

Theorem (Kuratowski)

A graph is non-planar iff it contains a subdivision of $K_{3,3}$ or K_5.





Problem set time!

Next times:

- More on planarity
- Graph colouring
- Ramsey Theory

Questions?