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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

Winter 2019 – February 12/13/14

Complex Numbers

Solutions

"Try it yourself" problems:

1. $(2i)^2 = -4$

2. $(-2i)^2 = -4$

3. $x = 5i$ or $-5i$

4. $(1+i) + (2i) = 1 + 3i$

5. $(10+7i) - (6+3i) = 4 + 4i$

6. $(1+i) \times (2i) = 2(i)^2 + 2i = -2 + 2i$

7. $$\begin{aligned}(10+7i) \times (6+3i) &= (10 \times 6) + (7i \times 3i) + (6 \times 7i) + (10 \times 3i) \\ &= 60 + 21(i)^2 + 42i + 30i \\ &= 60 - 21 + 42i + 30i \\ &= 39 + 72i\end{aligned}$$

8. $$\begin{aligned}(4+3i) \times (4-3i) &= (4 \times 4) + (3i \times (-3i)) + (4 \times (-3i)) + (4 \times 3i) \\ &= 16 - 9(i)^2 - 12i + 12i \\ &= 16 + 9 = 25\end{aligned}$$

9. $$\begin{aligned}(a+bi) \times (a-bi) &= (a \times a) + (bi \times (-bi)) + (a \times (-bi)) + (a \times bi) \\ &= a^2 - b^2(i)^2 - abi + abi \\ &= a^2 + b^2\end{aligned}$$

10. $\overline{(1+i)} = 1 - i$

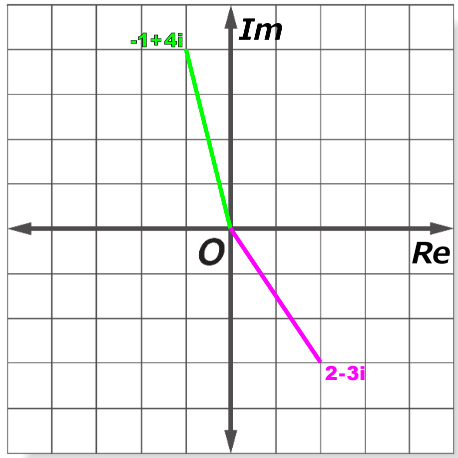
11.
$$\begin{aligned}\overline{(10+7i)\times(6+3i)} &= \overline{(10\times 6)+(7i\times 3i)+(10\times 3i)+(6\times 7i)}\\ &= \overline{60+21(i)^2+30i+42i}\\ &= \overline{60-21+30i+42i}\\ &= \overline{39+72i} = 39-72i\end{aligned}$$

12. $\overline{1}+\overline{i} = 1-i$

13.
$$\begin{aligned}\overline{(10+7i)}\times\overline{(6+3i)} &= (10-7i)\times(6-3i)\\ &= (10\times 6)+((-7i)\times(-3i))+(10\times(-3i))+(6\times(-7i))\\ &= 60+21(i)^2-30i-42i\\ &= 60-21-30i-42i\\ &= 39-72i\end{aligned}$$

14.
$$\begin{aligned}\frac{2+i}{i} &= \frac{2+i}{i}\times\frac{\overline{i}}{\overline{i}}\\ &= \frac{2+i}{i}\times\frac{-i}{-i}\\ &= \frac{(2\times(-i))+(i\times(-i))}{(i\times(-i))}\\ &= \frac{-2i+(-(i^2))}{-(i^2)}\\ &= \frac{1-2i}{1}\\ &= 1-2i\end{aligned}$$

15.
$$\begin{aligned}\frac{-7+3i}{4+5i} &= \frac{-7+3i}{4+5i}\times\frac{\overline{4+5i}}{\overline{4+5i}}\\ &= \frac{(-7+3i)\times(4-5i)}{(4+5i)\times(4-5i)}\\ &= \frac{((-7)\times 4)+(3i\times(-5i))+(3i\times 4)+((-7)\times(-5i))}{(4\times 4)+(5i\times(-5i))+(4\times 5i)+(4\times(-5i))}\\ &= \frac{-28-15(i)^2+12i+35i}{16-25(i^2)+20i-20i}\\ &= \frac{-28+15+47i}{16+25}\\ &= -\frac{13}{41}-\frac{47}{41}i\end{aligned}$$

- 

- The modulus of a complex number represents the length of the line segment from the origin to the point that number represents.

16.
$$\begin{aligned}|-1+4i| &= \sqrt{(-1+4i)\times\overline{(-1+4i)}}\\ &= \sqrt{(-1+4i)\times(-1-4i)}\\ &= \sqrt{((-1)\times(-1))+(4i\times(-4i))+((-1)\times(-4i))+((-1)\times 4i)}\\ &= \sqrt{1-16(i)^2+4i-4i}\\ &= \sqrt{1+16}=\sqrt{17}\approx 4.12\end{aligned}$$

17.
$$\begin{aligned}|2-3i| &= \sqrt{(2-3i)\times\overline{(2-3i)}}\\ &= \sqrt{(2-3i)\times(2+3i)}\\ &= \sqrt{(2\times 2)+((-3i)\times 3i)+(2\times 3i)+(2\times(-3i))}\\ &= \sqrt{4-9(i)^2+6i-6i}\\ &= \sqrt{4+9}=\sqrt{13}\approx 3.61\end{aligned}$$

End of lesson problems:

1. (a) $1+i-3i=1-2i$

   (b) $(5-2i)+(2+i)=5+2-2i+i=7-i$

   (c) $-2(7-8i)+3(3+2i)=-14+16i+9+6i=-5+22i$

   (d) $(2-i)+3(2+i)-2(1+3i)=2-i+6+3i-2-6i=6-4i$

   (e) $(6-8i)+2\overline{(2+5i)}=(6-8i)+2(2-5i)=6-8i+4-10i=10-18i$

(f)
$$\begin{aligned} 4(3+2i) - 5\overline{(1-i)} + 6\overline{(-3+i)} &= 12 + 8i - 5(1+i) + 6(-3-i) \\ &= 12 + 8i - 5 - 5i - 18 - 6i \\ &= -11 - 3i \end{aligned}$$

2. (a) $(3+i) \times 2i = (3 \times 2i) + (i \times 2i) = 6i + 2(i)^2 = -2 + 6i$

(b)
$$\begin{aligned} (-5+3i) \times (1+2i) &= ((-5) \times 1) + (3i \times 2i) + ((-5) \times 2i) + (1 \times 3i) \\ &= -5 + 6(i)^2 - 10i + 3i \\ &= -5 - 6 - 10i + 3i = -11 - 7i \end{aligned}$$

(c)
$$\begin{aligned} 4(2-3i) \times 2(1-4i) &= (8-12i) \times (2-8i) \\ &= (8 \times 2) + ((-12i) \times (-8i)) + (8 \times (-8i)) + (2 \times (-12i)) \\ &= 16 + 96(i)^2 - 64i - 24i \\ &= 16 - 96 - 64i - 24i \\ &= -80 - 88i \end{aligned}$$

(d)
$$\begin{aligned} (-1+2i) \times 2(5+2i) \times 3(1+6i) &= (-1+2i) \times (10+4i) \times (3+18i) \\ &= ((-10) + (2i \times 4i) + (-4i) + (10 \times 2i)) \times (3+18i) \\ &= (-10 + 8(i)^2 - 4i + 20i) \times (3+18i) \\ &= (-18+16i) \times (3+18i) \\ &= ((-18) \times 3) + (16i \times 18i) + ((-18) \times 18i) + (16i \times 3) \\ &= -54 + 288(i)^2 - 324i + 48i \\ &= -342 - 276i \end{aligned}$$

(e)
$$\begin{aligned} (-3-4i) \times 4\overline{(-3+i)} &= (-3-4i) \times 4(-3-i) \\ &= (-3-4i) \times (-12-4i) \\ &= ((-3)\times(-12)) + ((-4i)\times(-4i)) + ((-3)\times(-4i)) + ((-12)\times(-4i)) \\ &= 36 + 16(i)^2 + 12i + 48i \\ &= 36 - 16 + 12i + 48i = 20 + 60i \end{aligned}$$

(f)
$$2(4+i)\times -3\overline{(9-i)}\times\overline{(-1-i)} = (8+2i)\times -3(9+i)\times(-1+i)$$
$$= (8+2i)\times(-27-3i)\times(-1+i)$$
$$=((8\times(-27))+(2i\times(-3i))+(8\times(-3i))+((-27)\times 2i))\times(-1+i)$$
$$= (-216-6(i)^2-24i-54i)\times(-1+i)$$
$$= (-210-78i)\times(-1+i)$$
$$=((-210)\times(-1))+((-78i)\times i)+((-210)\times i)+((-1)\times(-78i))$$
$$= 210-78(i)^2-210i+78i$$
$$= 288-132i$$

3. (a)
$$(2+3i)\div i = \frac{2+3i}{i}\times\frac{\bar{i}}{\bar{i}} = \frac{(2+3i)\times(-i)}{i\times(-i)}$$
$$= \frac{-3(i)^2-2i}{-(i)^2} = \frac{3-2i}{1}$$
$$= 3-2i$$

(b)
$$(3-4i)\div(1+9i) = \frac{3-4i}{1+9i}\times\frac{\overline{1+9i}}{\overline{1+9i}} = \frac{3-4i}{1+9i}\times\frac{1-9i}{1-9i}$$
$$= \frac{(3\times 1)+((-4i)\times(-9i))+(3\times(-9i))+(1\times(-4i))}{(1\times 1)+(9i\times(-9i))+(1\times 9i)+(1\times(-9i))}$$
$$= \frac{3+36(i)^2-27i-4i}{1-81(i)^2+9i-9i}$$
$$= \frac{3-36-31i}{82} = -\frac{33}{82}-\frac{31}{82}i$$

(c)
$$-(1-7i)\div 3(4+2i) = \frac{-(1-7i)}{12+6i} = \frac{-(1-7i)}{12+6i}\times\frac{\overline{(12+6i)}}{\overline{(12+6i)}}$$
$$= \frac{(-1+7i)\times(12-6i)}{(12+6i)\times(12-6i)}$$
$$= \frac{((-1)\times 12)+(7i\times(-6i))+((-1)\times(-6i))+(12\times 7i)}{(12\times 12)+(6i\times(-6i))+(12\times 6i)+(12\times(-6i))}$$
$$= \frac{-12-42(i)^2+6i+84i}{144-36(i)^2+72i-72i}$$
$$= \frac{30+90i}{144+36} = \frac{30}{180}+\frac{90}{180}i = \frac{1}{6}+\frac{1}{2}i$$

(d)
$$
\begin{aligned}
((2-2i)\div(5+5i))\div 6(2-3i) &= \left(\frac{2-2i}{5+5i}\times\frac{\overline{5+5i}}{\overline{5+5i}}\right)\div(12-18i)\\
&= \left(\frac{2-2i}{5+5i}\times\frac{5-5i}{5-5i}\right)\div(12-18i)\\
&= \tfrac{(2\times 5)+((-2i)\times(-5i))+(2\times(-5i))+(5\times(-2i))}{(5\times 5)+(5i\times(-5i))+(5\times 5i)+(5\times(-5i))}\div(12-18i)\\
&= \frac{10+10(i)^2-10i-10i}{25-25(i)^2}\div(12-18i)\\
&= -\frac{20i}{50}\div(12-18i) = \frac{2i}{5(12-18i)}\\
&= \frac{2i}{60-90i}\times\frac{\overline{60-90i}}{\overline{60-90i}}\\
&= \frac{2i\times(60+90i)}{(60-90i)\times(60+90i)}\\
&= \frac{120i+180(i)^2}{(60\times 60)+(90i\times(-90i))+(60\times 90i)+(60\times(-90i))}\\
&= \frac{-180+120i}{3600+8100} = \frac{-180+120i}{11700}\\
&= -\frac{1}{65}+\frac{2}{195}i
\end{aligned}
$$

(e)
$$
\begin{aligned}
(1-2i)\div 2\overline{(4+3i)} &= \frac{1-2i}{2(4-3i)} = \frac{1-2i}{8-6i} = \frac{1-2i}{8-6i}\times\frac{\overline{8-6i}}{\overline{8-6i}}\\
&= \frac{1-2i}{8-6i}\times\frac{8+6i}{8+6i}\\
&= \frac{(1\times 8)+((-2i)\times 6i)+(1\times 6i)+((-2i)\times 8)}{(8\times 8)+((-6i)\times 6i)+(8\times 6i)+(8\times(-6i))}\\
&= \frac{8-12(i)^2+6i-16i}{64-36(i)^2+48i-48i}\\
&= \frac{20-10i}{100} = \frac{1}{5}-\frac{1}{10}i
\end{aligned}
$$

(f) $$\left(5(1+3i) \div 2\overline{(2-3i)}\right) \div -6\overline{(1-4i)} = \frac{5+15i}{\overline{4-6i}} \div \overline{(-6+24i)}$$

$$= \left(\frac{5+15i}{4+6i} \times \frac{\overline{4+6i}}{\overline{4+6i}}\right) \div (-6-24i)$$

$$= \frac{(5+15i)\times(4-6i)}{(4+6i)\times(4-6i)} \div (-6-24i)$$

$$= \frac{(5\times4)+(15i\times(-6i))+(5\times(-6i))+(4\times15i)}{(4\times4)+(6i\times(-6i))+(4\times6i)+(4\times(-6i))} \div (-6-24i)$$

$$= \frac{20-90(i)^2-30i+60i}{16-36(i)^2+24i-24i} \div (-6-24i)$$

$$= \frac{110+30i}{52} \div (-6-24i) = \frac{110+30i}{52(-6-24i)}$$

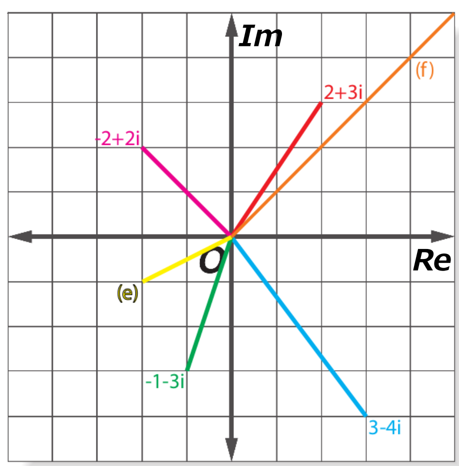
$$= \frac{110+30i}{-312-1248i} \times \frac{\overline{-312-1248i}}{\overline{-312-1248i}}$$

$$= \frac{(110+30i)\times(-312+1248i)}{(-312-1248i)\times(-312+1248i)}$$

$$= \frac{(110\times(-312))+(30i\times1248i)+(110\times1248i)+((-312)\times30i)}{((-312)\times(-312))+(1248i\times(-1248i))+((-312)\times(-1248i))+((-312)\times1248i)}$$

$$= \frac{-34320+37440(i)^2+137280i-9360i}{97344-1557504(i)^2+389376i-389376i}$$

$$= \frac{-71760+127920i}{1654748} = -\frac{17940}{413687} + \frac{31980}{413687}i$$

4. 

5. (a)

$$|2+3i| = \sqrt{(2+3i)\times\overline{(2+3i)}} = \sqrt{(2+3i)\times(2-3i)}$$

$$= \sqrt{(2\times2)+(3i\times(-3i))+(2\times3i)+(2\times(-3i))}$$

$$= \sqrt{4-9(i)^2+6i-6i}$$

$$= \sqrt{4+9} = \sqrt{13} \approx 3.606$$

(b)
$$\begin{aligned}|3-4i| &= \sqrt{(3-4i)\times\overline{(3-4i)}} = \sqrt{(3-4i)\times(3+4i)} \\ &= \sqrt{(3\times 3)+((-4i)\times 4i)+(3\times(-4i))+(3\times 4i)} \\ &= \sqrt{9-16(i)^2+12i-12i} \\ &= \sqrt{9+16} = \sqrt{25} = 5\end{aligned}$$

(c)
$$\begin{aligned}|-1-3i| &= \sqrt{(-1-3i)\times\overline{(-1-3i)}} = \sqrt{(-1-3i)\times(-1+3i)} \\ &= \sqrt{((-1)\times(-1))+((-3i)\times 3i)+((-1)\times 3i)+((-1)\times(-3i))} \\ &= \sqrt{1-9(i)^2+3i-3i} \\ &= \sqrt{1+9} = \sqrt{10} \approx 3.162\end{aligned}$$

(d)
$$\begin{aligned}|-2+2i| &= \sqrt{(-2+2i)\times\overline{(-2+2i)}} = \sqrt{(-2+2i)\times(-2-2i)} \\ &= \sqrt{((-2)\times(-2))+(2i\times(-2i))+((-2)\times(-2i))+((-2)\times 2i)} \\ &= \sqrt{4-4(i)^2+4i-4i} \\ &= \sqrt{4+4} = \sqrt{8} \approx 2.828\end{aligned}$$

(e)
$$\begin{aligned}|2(1-2i)-(4-3i)| &= |(2-4i)+(-4+3i)| = |-2-i| \\ &= \sqrt{(-2-i)\times\overline{(-2-i)}} = \sqrt{(-2-i)\times(-2+i)} \\ &= \sqrt{((-2)\times(-2))+(i\times(-i))+((-2)\times i)+((-2)\times(-i))} \\ &= \sqrt{4-(i)^2+2i-2i} \\ &= \sqrt{4+1} = \sqrt{5} \approx 2.236\end{aligned}$$

(f)
$$\begin{aligned}|(1+3i)\times(2-i)| &= |(1\times 2)+(3i\times(-i))+(1\times(-i))+(2\times 3i)| \\ &= |2-3(i)^2-i+6i| = |5+5i| \\ &= \sqrt{(5+5i)\times\overline{(5+5i)}} = \sqrt{(5+5i)\times(5-5i)} \\ &= \sqrt{(5\times 5)+(5i\times(-5i))+(5\times 5i)+(5\times(-5i))} \\ &= \sqrt{25-25(i)^2+25i-25i} \\ &= \sqrt{25+25} = \sqrt{50} \approx 7.071\end{aligned}$$

6. "LHS" is short for **l**eft **h**and **s**ide and "RHS" is short for **r**ight **h**and **s**ide.

   (a) **Proof:**
   Let $z = a + bi$ and $w = c + di$ where a, b, c, and d are all real numbers. Then,

$$\begin{aligned} \text{LHS} = \overline{z + w} &= \overline{(a + bi) + (c + di)} \\ &= \overline{(a + c) + (b + d)i} \\ &= (a + c) - (b + d)i \end{aligned} \qquad \begin{aligned} \text{RHS} = \overline{z} + \overline{w} &= \overline{(a + bi)} + \overline{(c + di)} \\ &= (a - bi) + (c - di) \\ &= (a + c) - (b + d)i \end{aligned}$$

   Hence LHS = RHS so the statement is true. □

   (b) **Proof:**
   Again, let $z = a + bi$ and $w = c + di$ where a, b, c, and d are all real numbers. Then,

$$\begin{aligned} \text{LHS} = \overline{z \times w} &= \overline{(a + bi) \times (c + di)} \\ &= \overline{(ac - bd) + (ad + bc)i} \\ &= (ac - bd) - (ad + bc)i \end{aligned} \qquad \begin{aligned} \text{RHS} = \overline{z} \times \overline{w} &= \overline{(a + bi)} \times \overline{(c + di)} \\ &= (a - bi) \times (c - di) \\ &= (ac - bd) - (ad + bc)i \end{aligned}$$

   Hence LHS = RHS so the statement is true. □

   (c) **Proof:**
   Again, let $z = a + bi$ and $w = c + di$ where a, b, c, and d are all real numbers where c and d are nonzero. Then,

$$\begin{aligned} \text{LHS} = \overline{z \div w} &= \overline{(a + bi) \div (c + di)} \\ &= \overline{\left(\frac{a + bi}{c + di} \times \frac{c - di}{c - di}\right)} \\ &= \overline{\frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i} \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i \end{aligned} \qquad \begin{aligned} \text{RHS} = \overline{z} \div \overline{w} &= \overline{(a + bi)} \div \overline{(c + di)} \\ &= (a - bi) \div (c - di) \\ &= \frac{a - bi}{c - di} \times \frac{c + di}{c + di} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{ad - bc}{c^2 + d^2}i \\ &= \frac{ac + bd}{c^2 + d^2} - \frac{bc - ad}{c^2 + d^2}i \end{aligned}$$

   Hence LHS = RHS so the statement is true. □

7. (a) Here $a = 1$, $b = 0$, and $c = -1$. Using the quadratic formula, $x = \frac{-(0)\pm\sqrt{(0)^2-4(1)(-1)}}{2(1)} = +1$ or -1.

   (b) Here $a = 1$, $b = 2$, and $c = 1$. Using the quadratic formula, $x = \frac{-(2)\pm\sqrt{(2)^2-4(1)(1)}}{2(1)} = \frac{-2\pm\sqrt{4-4}}{2} = -1$.

   (c) Here $a = 2$, $b = -6$, and $c = -8$. Using the quadratic formula, $x = \frac{-(-6)\pm\sqrt{(-6)^2-4(2)(-8)}}{2(2)} = \frac{6\pm\sqrt{36+64}}{4} = \frac{6\pm10}{4} = 4$ or -1.

   (d) Here $a = 1$, $b = 0$, and $c = 1$. Using the quadratic formula, $x = \frac{-(0)\pm\sqrt{(0)^2-4(1)(1)}}{2(1)} = \frac{\pm\sqrt{-4}}{2} = +i$ or $-i$.

   (e) Here $a = 1$, $b = -1$, and $c = 4$. Using the quadratic formula, $x = \frac{-(-1)\pm\sqrt{(-1)^2-4(1)(4)}}{2(1)} = \frac{1\pm\sqrt{1-16}}{2} = \frac{1}{2} + \frac{\sqrt{15}}{2}i$ or $\frac{1}{2} - \frac{\sqrt{15}}{2}i$.

   (f) Here $a = 3$, $b = 2$, and $c = -8$. Using the quadratic formula, $x = \frac{-(2)\pm\sqrt{(2)^2-4(3)(-8)}}{2(3)} = \frac{-2\pm\sqrt{4-96}}{6} = -\frac{1}{3} + \frac{\sqrt{92}}{6}i$ or $-\frac{1}{3} - \frac{\sqrt{92}}{6}i$.