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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

Winter 2019 – Feb. 19, 20, & 21

Exponentiation

Solutions

Blanks (in order of occurrence): *exponentiation*, *square root*, *one*, $\sqrt[3]{64} = 4$

"Try it yourself" #1:

1. $4^3 = 4 \times 4 \times 4 = 64$
2. $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$
3. $8^2 = 8 \times 8 = 64$

"Try it yourself" #2:

1. $1^2 = 1 \times 1 = 1$
2. $1^3 = 1 \times 1 \times 1 = 1$
3. $1^{2019} = \underbrace{1 \times \cdots \times 1}_{\text{2019 times}} = 1$
4. $0^2 = 0 \times 0 = 0$
5. $0^3 = 0 \times 0 \times 0 = 0$
6. $0^{2019} = \underbrace{0 \times \cdots \times 0}_{\text{2019 times}} = 0$

"Try it yourself" #3:

1. $2^1 = 2$
2. $3^1 = 3$
3. $2019^1 = 2019$

"Try it yourself" #4:

1. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ or 0.125
2. $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ or 0.04
3. $10^{-4} = \frac{1}{10^4} = \frac{1}{10000}$ or 0.0001

"Try it yourself" #5:

1. $\sqrt{49} = 7$ since $7^2 = 49$
2. $\sqrt{81} = 9$ since $9^2 = 81$
3. $\sqrt[3]{-27} = -3$ since $(-3)^3 = -27$
4. $\sqrt[5]{32} = 2$ since $2^5 = 32$

"Try it yourself" #6:

1. $49^{\frac{1}{2}} = \sqrt{49} = 7$
2. $81^{\frac{1}{2}} = \sqrt{81} = 9$
3. $(-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3$
4. $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$

"Try it yourself" #7:

1. $2^2 \times 2^2 = 2^4 = 16$
2. $3^2 \times 3 = 3^3 = 27$
3. $5^2 \times 5^2 = 5^4 = 625$
4. $6^1 \times 6^{-1} = 6^0 = 1$

"Try it yourself" #8:

1. $(2^2)^2 = 2^4 = 16$
2. $(3^4)^{\frac{1}{2}} = 3^2 = 9$
3. $(5^2)^{-1} = \frac{1}{5^2} = \frac{1}{25} = 0.04$
4. $(6^1)^{-2} = \frac{1}{6^2} = \frac{1}{36} = 0.02\overline{7}$

"Try it yourself" #9: In numerical order, the expressions evaluate to 2, 2.25, $2.\overline{370}$, 2.692, 2.705, and 2.718.

"Try it yourself" #10: We identify our variables: $P = \$400000$, $r = 5\% = 0.05$, $n = 12$ since interest compounds monthly, and $t = 3$ years. Assembling the equation,

$$\begin{aligned} A = P\left(1 + \frac{r}{n}\right)^{nt} &= 400000\left(1 + \frac{0.05}{12}\right)^{(12\times 3)} \\ &= 400000(1 + 0.0041\overline{6})^{36} \\ &= 400000 \times 1.16147223133 \\ &= \$464\ 588.89. \end{aligned}$$

Hence after three years, we would have \$464 588.89.

"Try it yourself" #11:

1. We move the decimal point forward 3 places to get 3.52×10^3.
2. We move the decimal point forward 9 places to get 1.1658×10^9.
3. We move the decimal point back 3 places to get 3.65×10^{-3}.
4. We move the decimal point back 5 places to get 9.4651×10^{-5}.

"Try it yourself" #12:

1. There are 3.14496×10^7 seconds in a year and approximately 3.671×10^7 people in Canada. Since they are both multiples of 10^7, they are of the same order of magnitude.

2. The (estimated) population of the world is about 7.7×10^9, while there is only one star in the solar system—the sun! Obviously, these number are not the same order of magnitude.

3. The number of pixels in an HD (1920×1080) screen is 2.0736×10^6, while the moon is approximately 3.844×10^5 km away from the earth on average. Since these numbers are not multiples of the same power of ten, they are of different orders of magnitude.

4. The human hair is, at thickest, 1.81×10^{-6} m, while a the smallest grains of sand have diameters of 7.4×10^{-5} m. Hence the human hair is typically at least one order of magnitude smaller than a grain of sand.

End-of-lesson problems:

1. (a) $5^2 = 5 \times 5 = 25$
   (b) $3^3 = 3 \times 3 \times 3 = 27$
   (c) $4^{-1} = \frac{1}{4}$ or 0.25
   (d) $9^{\frac{1}{2}} = \sqrt{9} = 3$
   (e) $64^{-\frac{1}{3}} = \frac{1}{64^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$ or 0.25
   (f) $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{\sqrt[3]{64}} = \frac{1}{4}$ or 0.25

2. (a) $2^2 \times 2^3 = 2^{2+3} = 2^5$
   (b) $3^8 \times 3^{-6} = 3^{8-6} = 3^2$
   (c) $\sqrt{5} \times 5^{\frac{3}{2}} = 5^{\frac{1}{2}} \times 5^{\frac{3}{2}} = 5^{\frac{1}{2}+\frac{3}{2}} = 5^2$
   (d) $8^{19} \times \frac{1}{8^{20}} = 8^{19} \times 8^{-20} = 8^{-1}$ or $\frac{1}{8}$
   (e) $\left(11^2 \times \frac{1}{\sqrt[3]{11}} \div 11^{-\frac{2}{3}}\right) = \left(11^2 \times \frac{1}{11^{\frac{1}{3}}} \div \frac{1}{11^{\frac{2}{3}}}\right) = \left(11^2 \times 11^{-\frac{1}{3}} \times 11^{\frac{2}{3}}\right) = 11^{\frac{6}{3}-\frac{1}{3}+\frac{2}{3}} = 11^{\frac{7}{3}} \approx 269.102$

3. (a) $(2^2)^{\frac{1}{2}} = 2^{\left(2\times\frac{1}{2}\right)} = 2^1 = 2$
   (b) $(3^{-1})^{-3} = 3^{((-1)\times(-3))} = 3^3 = 27$
   (c) $\left(\left(\frac{1}{5}\right)^2\right)^{-\frac{3}{2}} = \left((5^{-1})^2\right)^{-\frac{3}{2}} = \left(5^{((-1)\times 2)}\right)^{-\frac{3}{2}} = (5^{-2})^{-\frac{3}{2}} = 5^{\left((-2)\times(-\frac{3}{2})\right)} = 5^3 = 125$
   (d) $\left(8^{\frac{1}{3}}\right)^{\sqrt[3]{27}} = \left(8^{\frac{1}{3}}\right)^3 = 8^{\left(\frac{1}{3}\times 3\right)} = 8^1 = 8$
   (e) $\left(\left(\frac{1}{\sqrt{11}}\right)^{\frac{1}{4}}\right)^{-\sqrt{16}} = \left(\left(\frac{1}{11^{\frac{1}{2}}}\right)^{\frac{1}{4}}\right)^{-\sqrt{16}} = \left(\left(11^{-\frac{1}{2}}\right)^{\frac{1}{4}}\right)^{-\sqrt{16}} = \left(11^{\left((-\frac{1}{2})\times\frac{1}{4}\right)}\right)^{-\sqrt{16}} = \left(11^{-\frac{1}{8}}\right)^{-\sqrt{16}} = \left(11^{-\frac{1}{8}}\right)^{-4} = 11^{\left((-\frac{1}{8})\times(-4)\right)} = 11^{\frac{1}{2}}$ or $\sqrt{11} \approx 3.317$

4. (a) $10^x - 10 = 9990 \implies 10^x = 10000$, so $x = 4$
   (b) $4^x = 64^2 \implies 4^x = (4^3)^2 \implies 4^x = 4^6$, so $x = 6$
   (c) $3^6 = 27^x \implies 3^6 = (3^3)^x \implies 3^6 = 3^{3x} \implies 3x = 6$, so $x = 2$
   (d) $5^{33} = 125^x \implies 5^{33} = (5^3)^x \implies 5^{33} = 5^{3x} \implies 3x = 33$, so $x = 11$
   (e) $6^{x+2} = 216 \implies 6^{x+2} = 6^3 \implies x + 2 = 3$, so $x = 1$
   (f) $8^{x-1} = 2^6 \implies (2^3)^{x-1} \implies 2^6 = 2^{3x-3} \implies 3x - 3 = 6$, so $x = 3$

5. (a) Here we recognize $t = 1$ year, $r = 1\% = 0.01$, $P = \$500$, and $n = 12$ times per year. Plugging into the formula, we get that $\$500\left(1 + \frac{0.1}{12}\right)^{12} = \$552.36$
   (b) Here we recognize $t = 5$ years, $r = 2\% = 0.02$, $P = \$1500$, and $n = 4$ times per year. Plugging into the formula, we get that $\$1500\left(1 + \frac{0.02}{4}\right)^{(4\times5)} = \$1\ 657.34$
   (c) Here we recognize $t = 10$ years, $r = 4\% = 0.04$, $P = \$1000000$, and $n = 1$ time per year. Plugging into the formula, we get that $\$1000000\left(1 + \frac{0.04}{1}\right)^{10} = \$1\ 480\ 244.28$
   (d) Here we recognize $t = 3$ years, $r = 1.8\% = 0.018$, $P = \$5000$, and $n = 52$ times per year. Plugging into the formula, we get that $\$5000\left(1 + \frac{0.018}{52}\right)^{(52\times3)} = \$5\ 277.37$
   (e) Here we recognize $t = 7$ years, $r = 2.7\% = 0.027$, $P = 2001$, and $n = 12$. Plugging into the formula, we get that $\$2001\left(1 + \frac{0.027}{12}\right)^{(7\times12)} = \$2\ 416.78$

6. We recognize \$10 000 000 as Jimmy Bob's principal, 0.015 or 1.5% as the interest rate, and monthly or 12 times per year to be the compounding period. The duration Jimmy Bob lets his money accumulate interest is one year. We now plug into the formula to get that
$$\$10000000\left(1 + \frac{0.015}{12}\right)^{12} = \$1151035.56.$$
We want what Jimmy Bob would have to live off of, so we subtract the money he would leave in his account: $1151035.56 - 1000000 = 151035.56$. So Jimmy Bob would have \$151 035.56 to live on every year.

7. (a) The two differ by a whopping **five** orders of magnitude. This means that, on average, 10 000 stars would need to be born every second for the entire duration of the universe to get to where we are now.
   (b) The two differ by one order of magnitude. It is worth noting that the approximate "size" of the electron compared to a metre is the same as comparing a millimetre to the distance to Uranus.

(c) The two are the same order of magnitude! Think about that! The thinnest spider silk is actually thinner than a **cell**. One point of clarification—most spider silk is not that thin. The silk that we can see is typically much thicker, or coated in particles that make it **look** thicker.

(d) The two differ by two orders of magnitude. In this case, the idea of order of magnitude isn't strictly correct. While the two **are** two orders of magnitude apart, they do not differ by a factor of 100—a collection of only 24 elephants would have about the same mass as a blue whale. That said, we often think of elephants has huge beasts, but blue whales are the **largest animals to have ever lived**. Their hearts weigh as much as two average humans, and we could sit in one comfortably.

(e) The Burj Khalifa in Dubai is well over the height of **400 humans** stacked on top of one another. In order to make a human pyramid this tall, we would need at least 80200 people, and that's not taking into account that people would be standing on each other's shoulders!

(f) The two differ by one order of magnitude. As tall as the Burj Khalifa is, you could fit eleven of them inside Mount Everest, and nearly **14** of them in the deepest part of the Mariana Trench.

8. The only two distinct integers that satisfy this equation are 2 and 4.