UNIVERSITY OF
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Faculty of Mathematics

Waterloo, Ontario

Centre for Education in

Mathematics and Computing

Grade 7/8 Math Circles

Winter 2019 – Feb 26/27/28

Solution Set - The Pythagorean Theorem

"Prove the equations" (pg. 7) solutions:

1. $c^2 = a^2 + b^2 \;\rightarrow\; \sqrt{c^2} = \sqrt{a^2 + b^2} \;\rightarrow\; c = \sqrt{a^2 + b^2}$

2. $a^2 + b^2 = c^2 \;\rightarrow\; a^2 = c^2 - b^2 \;\rightarrow\; \sqrt{a^2} = \sqrt{c^2 - b^2} \;\rightarrow\; a = \sqrt{c^2 - b^2}$

3. $a^2 + b^2 = c^2 \;\rightarrow\; b^2 = c^2 - a^2 \;\rightarrow\; \sqrt{b^2} = \sqrt{c^2 - a^2} \;\rightarrow\; b = \sqrt{c^2 - a^2}$

Page 8 solutions:

1.

$$
\begin{aligned}
a &= \sqrt{15^2 - 12^2} \\
&= \sqrt{225 - 144} \\
&= \sqrt{81} \\
&= 9
\end{aligned}
$$

2.

$$
\begin{aligned}
c &= \sqrt{8^2 + 15^2} \\
&= \sqrt{64 + 225} \\
&= \sqrt{289} \\
&= 17
\end{aligned}
$$

3.

$$
\begin{aligned}
c &= \sqrt{1^2 + 1^2} \\
&= \sqrt{1 + 1} \\
&= \sqrt{2} \\
&= 1.4142135...
\end{aligned}
$$

Page 10 solutions:

1. The lengths of the sides of the triangle defined by these two points are

   $|(-1) - 0| = 1$ and $|5 - 3| = 2$, so the distance between the points is
   $\sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.236$.

2. The lengths of the sides of the triangle defined by these two points are

   $|2 - 3| = 1$ and $|(-6) - 1| = 7$, so the distance between the points is
   $\sqrt{1^2 + 7^2} = \sqrt{1 + 49} = \sqrt{50} \approx 7.071$.

3. The lengths of the sides of the prism defined by these two points are

   $|0 - 3| = 3$, $|1 - 6| = 5$, and $|1 - 0| = 1$, so the distance between the points is
   $\sqrt{3^2 + 5^2 + 1^2} = \sqrt{9 + 25 + 1} = \sqrt{35} \approx 5.916$.

4. The lengths of the sides of the prism defined by these two points are

   $|4 - (-5)| = 9$, $|1 - 6| = 5$, and $|(-3) - (-1)| = 2$

   , so the distance between the points is
   $\sqrt{9^2 + 5^2 + 2^2} = \sqrt{81 + 25 + 4} = \sqrt{110} \approx 10.488$.

5. The lengths of the sides of the prism defined by these two points are

$$|1 - 3| = 2, \ |7 - (-4)| = 11, \ |(-3) - (-6)| = 3, \text{ and } |2 - (-2)| = 4,$$

so the distance between the points is

$$\sqrt{2^2 + 11^2 + 3^2 + 4^2} = \sqrt{4 + 121 + 9 + 16} = \sqrt{150} \approx 12.247.$$

End-of-lesson problems:

1. Let a and b be the lengths of the legs of the right triangles below (in cm), and let c be the length of the hypotenuse (in cm). Solve for the missing length. Simplify your answers.

   (a) $a = 1$, $b = \sqrt{3}$, $c =$?

$$\begin{aligned} c &= \sqrt{1^2 + \sqrt{3}^2} \\ &= \sqrt{1 + 3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

   (b) $a = 2$, $b = 2$, $c =$?

$$\begin{aligned} c &= \sqrt{2^2 + 2^2} \\ &= \sqrt{4 + 4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

(c) $a = 6$, $b =$?, $c = 10$

$$\begin{aligned} b &= \sqrt{10^2 - 6^2} \\ &= \sqrt{100 - 36} \\ &= \sqrt{64} \\ &= 8 \end{aligned}$$
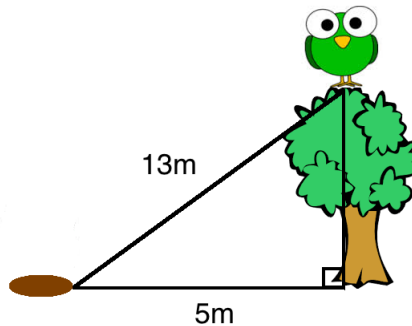
(d) $a =$?, $b = 24$, $c = 25$

$$\begin{aligned} a &= \sqrt{25^2 - 24^2} \\ &= \sqrt{625 - 576} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

(e) $a = 2$, $b = 4$, $c =$?

$$\begin{aligned} c &= \sqrt{2^2 + 4^2} \\ &= \sqrt{4 + 16} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

2. If a bird is on the ground, 5 metres away from the base of a tree, and the bird flies 13 metres to the top of the tree, how tall is the tree?

Let a be the distance in between the bird on the ground and the base of the tree (5m). Let b be the height of the tree, and let c be the distance the bird flew to get to the top of the tree (13m). Then, we have the following picture:

Pictures from: http://www.i2clipart.com/search/bird/1
and http://www.clipartpanda.com/categories/tree-clip-art-background

This forms a right triangle, which allows us to solve for the height of the tree using the Pythagorean Theorem. So we have,

$$
\begin{aligned}
b &= \sqrt{13^2 - 5^2} \\
&= \sqrt{169 - 25} \\
&= \sqrt{144} \\
&= 12
\end{aligned}
$$

Therefore, the tree is 12m tall.

3. Denver is in a hot air balloon that has just taken off and is now floating 12 metres above its launching point. Regan is standing on the ground, 16 metres away from the launching point. How far apart are Denver and Regan?

Let a be the height to which Denver has travelled (12m). Let b be the distance between the Regan and the launching point (16m), and let c be the distance between Denver and Regan. Then, we have the following picture:

12

16

Picture from: http://www.clker.com/clipart-hot-air-balloon-8.html
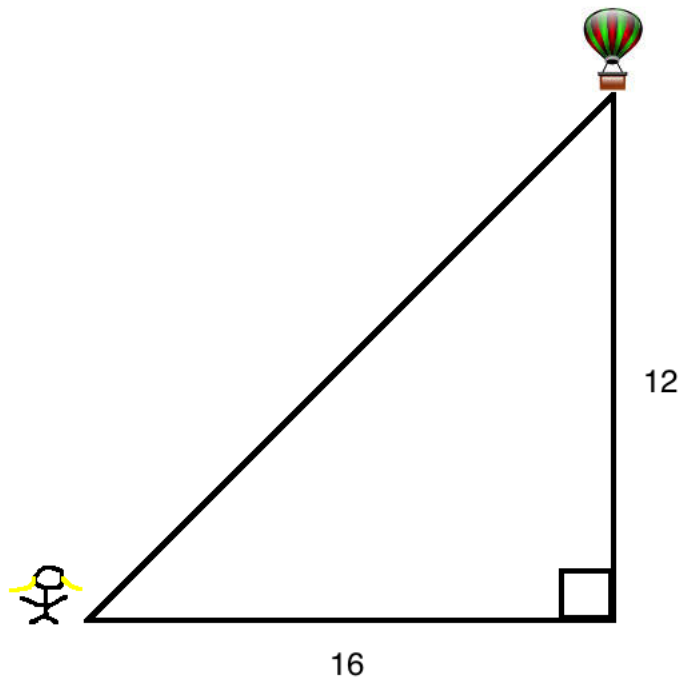
This forms a right triangle, which allows us to solve for how far apart Denver and Regan are using the Pythagorean Theorem. So we have,

$$
\begin{aligned}
c &= \sqrt{12^2 + 16^2} \\
&= \sqrt{144 + 256} \\
&= \sqrt{400} \\
&= 20
\end{aligned}
$$

Therefore, Denver and Regan are 20m apart.

4. The Manhattan distances are...

   (a) $4 - 4 = 0$ and $3 - 1 = 2$, so the Manhattan distance is 2.

   (b) $3 - 1 = 2$ and $4 - 2 = 2$, so the Manhattan distance is 4.

   (c) Counting, the Manhattan distance is 10.

   (d) Counting, the Manhattan distance is 8.

   (e) Counting, the Manhattan distance is 14.

   (f) Counting, the Manhattan distance is 16.

   These can be found by counting the whole number units along the grid joining the two points. Alternatively, you can find the difference in each point's co-ordinates, then add the two differences.

5. Your graph should look something like this:
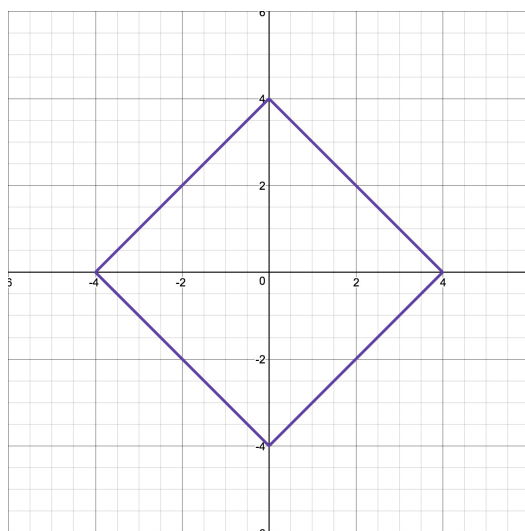
   Why? Because all the following points have a Manhattan distance from the origin of 4: $(4, 0)$, $(3, 1)$, $(2, 2)$, $(1, 3)$, $(0, 4)$, $(-1, 3)$, $(-2, 2)$, $(-3, 1)$, $(-4, 0)$, $(-3, -1)$, $(-2, -2)$, $(-1, -3)$, $(0, -4)$, $(1, -3)$, $(2, -2)$, and $(3, -1)$. You can then join the lines the points form.

6. If the length of a rectangle is 24m and the length of the diagonal is 26m, what is the width of the rectangle.

   Let a be the length in metres, let b be the width in metres, and let c be the length of the diagonal in metres. Then, we have the following picture:
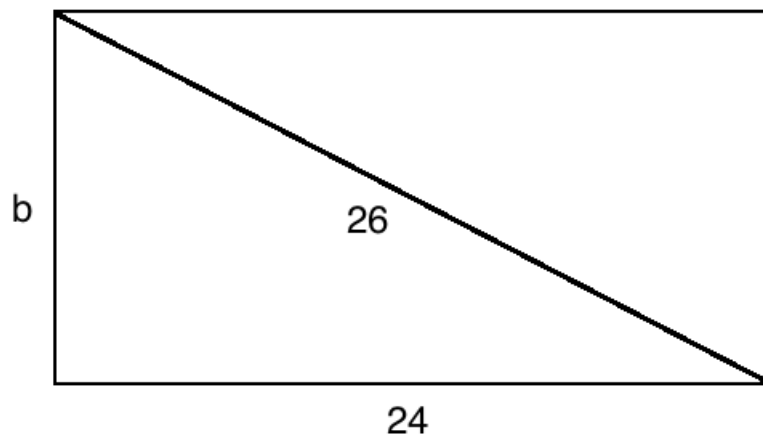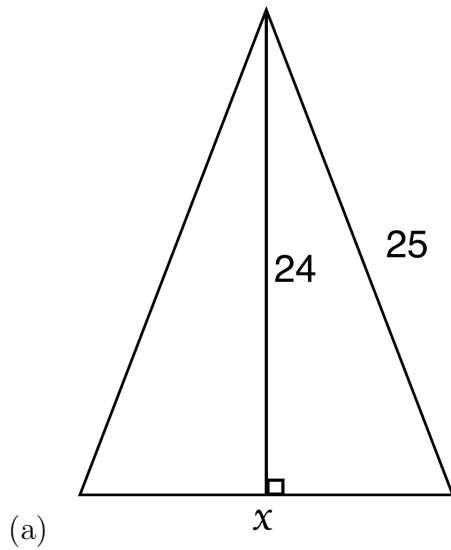
All we have to do now is solve for b using the Pythagorean Theorem:

$$
\begin{aligned}
b &= \sqrt{26^2 - 24^2} \\
&= \sqrt{676 - 576} \\
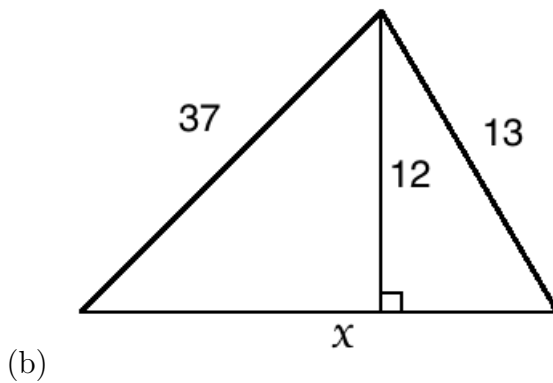&= \sqrt{100} \\
&= 10
\end{aligned}
$$

Therefore, the width of the rectangle is 10m.

7. Solve for the missing lengths in centimetres.

(a)

Using the lengths given, we can solve for $\frac{1}{2}$ the length of x. Therefore,

$$
\begin{aligned}
x &= 2\sqrt{25^2 - 24^2} \\
&= 2\sqrt{625 - 576} \\
&= 2\sqrt{49} \\
&= 2(7) \\
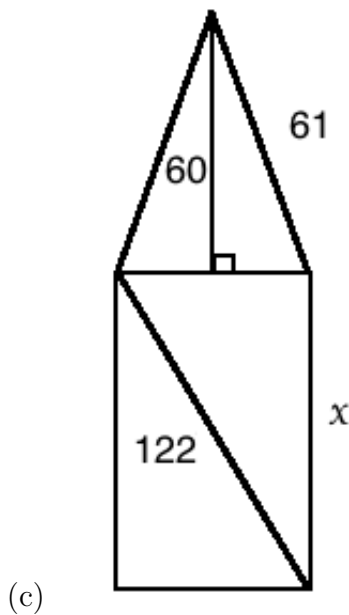&= 14cm
\end{aligned}
$$

(b)

To solve for x, we'll solve for the length of the base of the two right triangles. For the

base of the right triangle on the left (call thing length a):

$$
\begin{aligned}
a &= \sqrt{37^2 - 12^2} \\
&= \sqrt{1,369 - 144} \\
&= \sqrt{1,225} \\
&= 35cm
\end{aligned}
$$

For the right triangle on the right (call thing length b):

$$
\begin{aligned}
b &= \sqrt{13^2 - 12^2} \\
&= \sqrt{169 - 144} \\
&= \sqrt{25} \\
&= 5cm
\end{aligned}
$$

Since $x = a + b$, this tell us that $x = 40cm$

(c)

We'll first solve for the width of the rectangle. Call this w. Then,

$$
\begin{aligned}
w &= 2\sqrt{61^2 - 60^2} \\
&= 2\sqrt{3,721 - 3600} \\
&= 2\sqrt{121} \\
&= 2(11) \\
&= 22cm
\end{aligned}
$$

Now we can solve for x using the Pythagorean Theorem again. But first, notice that $22 = 2(11)$ and $122 = 2(61)$. This tell us that the bigger triangle is just a multiple of the Pythagorean Triple we just worked with (11 60 61). So the length of x must be $2(60) = 120$cm.

8. Solve for the length of AB
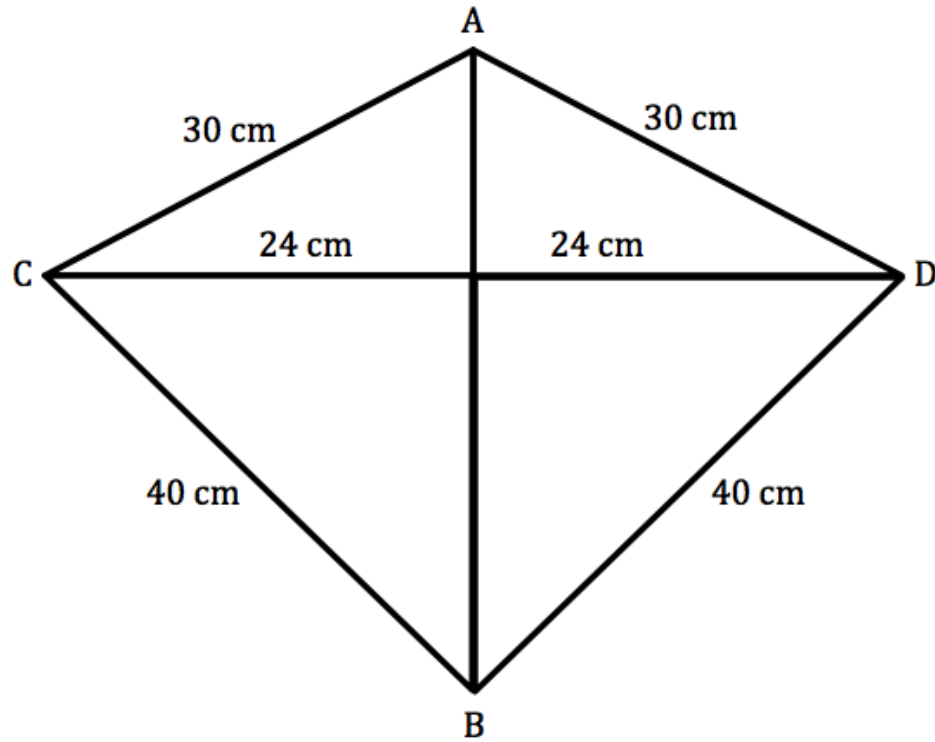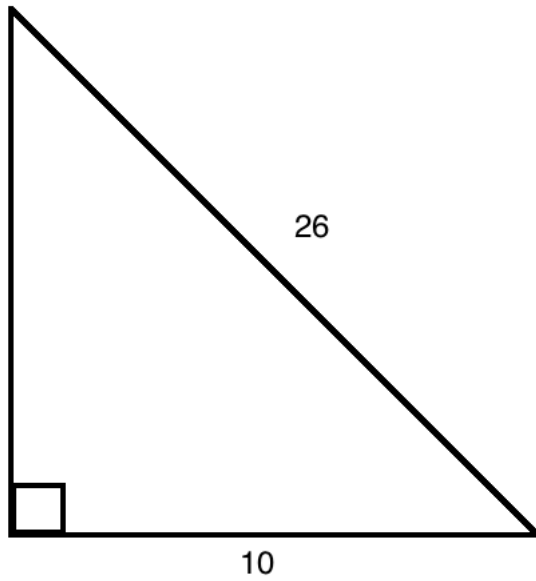

For the top portion of AB (call this length x):

$$\begin{aligned} x &= \sqrt{30^2 - 24^2} \\ &= \sqrt{900 - 576} \\ &= \sqrt{324} \\ &= 18cm \end{aligned}$$

For the bottom portion of AB (call this length y):

$$\begin{aligned} y &= \sqrt{40^2 - 24^2} \\ &= \sqrt{1600 - 576} \\ &= \sqrt{1024} \\ &= 32cm \end{aligned}$$

Since the length of AB is just $x + y$, its length is 50cm.

9. Solve for the area of the following shapes (Assume all lengths are in centimetres):
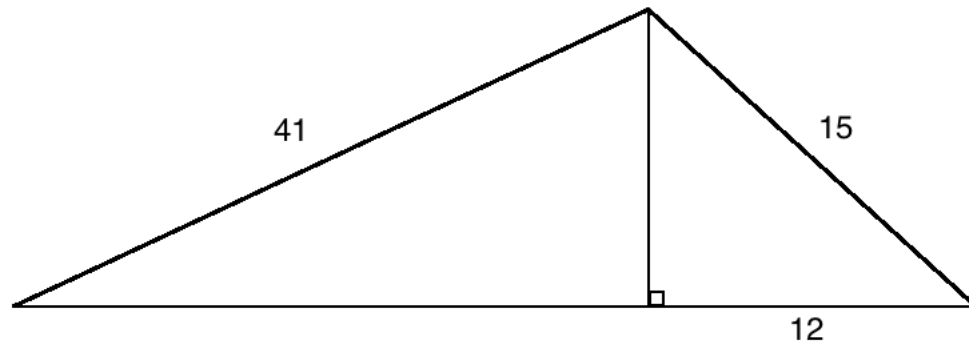

(a)

First we must solve for the height of this triangle. Call this h. Then,

$$
\begin{aligned}
h &= \sqrt{26^2 - 10^2} \\
&= \sqrt{676 - 100} \\
&= \sqrt{576} \\
&= 24cm
\end{aligned}
$$

Now for the area of the triangle:
$$
\begin{aligned}
A &= \frac{b * h}{2} \\
&= \frac{10 * 24}{2} \\
&= \frac{240}{2} \\
&= 120cm^2
\end{aligned}
$$

(b)

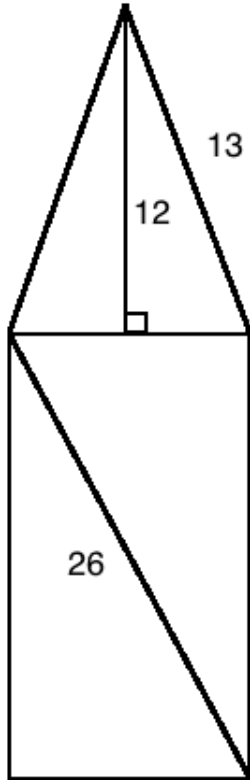First we must solve for the height of this triangle. Call this h. Then,

$$
\begin{aligned}
h &= \sqrt{15^2 - 12^2} \\
&= \sqrt{225 - 144} \\
&= \sqrt{81} \\
&= 9cm
\end{aligned}
$$

Now let's solve for the base of the right triangle on the left. Call this b_2. Then,

$$
\begin{aligned}
b_2 &= \sqrt{41^2 - 9^2} \\
&= \sqrt{1681 - 81} \\
&= \sqrt{1600} \\
&= 40cm
\end{aligned}
$$

Then, combining our two bases we find the area of this triangle:

$$
\begin{aligned}
A &= \frac{b * h}{2} \\
&= \frac{(40 + 12) * 9}{2} \\
&= \frac{468}{2} \\
&= 234cm^2
\end{aligned}
$$

(c)

First we must solve for the width of the rectangle. Call this w. Then,

$$
\begin{aligned}
w &= 2\sqrt{13^2 - 12^2} \\
&= 2\sqrt{169 - 144} \\
&= 2\sqrt{25} \\
&= 2(5) \\
&= 10cm
\end{aligned}
$$

Note that the width of our rectangle is also the base of our triangle. Now we must solve for the length of our rectangle. But, notice that $2(5) = 10$ and $2(13) = 26$. So the bigger triangle is just a multiple of the Pythagorean Triple from the triangle above (5 12 13). So the length must be $2(12) = 24$. The area of this figure is then,

$$
\begin{aligned}
A &= \frac{10 * 12}{2} + 10(24) \\
&= \frac{120}{2} + 240 \\
&= 60 + 240 \\
&= 300cm^2
\end{aligned}
$$

10. If the area of of an isosceles triangle is $32cm^2$, and the base and height of the triangle are the same, what are the lengths of the other two sides.
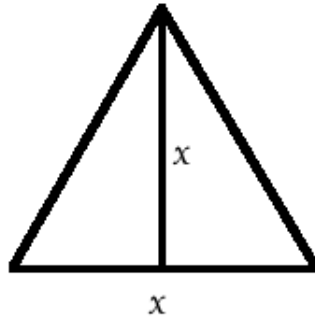
Let x be the base and height of the isosceles triangle. Then, using the area, we can solve for x:

$$\frac{x * x}{2} = 32$$
$$\rightarrow x^2 = 64$$
$$\rightarrow \sqrt{x^2} = \sqrt{64}$$
$$\rightarrow x = 8cm$$

Since this is an isosceles triangle, both bases of the two right triangles must be half the length of x (4cm). We then solve for the hypotenuse of one of the right triangles, because this will give us the length of both (isosceles triangle). Call this length c. Then,

$$c = \sqrt{4^2 + 8^2}$$
$$= \sqrt{16 + 64}$$
$$= \sqrt{80}$$
$$= \sqrt{16 * 5}$$
$$= 4\sqrt{5}cm$$

11. In the proof of the Pythagorean Theorem, use algebra to show that the area of the shaded region of the big square is equal to the area of the shaded region of the two smaller squares.

The total area of the figure in its first configuration is

$$A = c^2 + 2ab$$

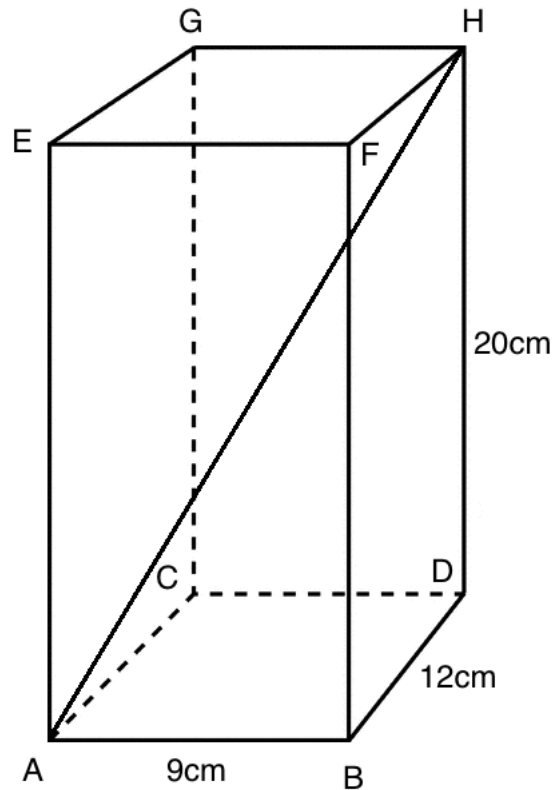The total area of the figure in its second configuration is

$$A = a^2 + b^2 + 2ab$$

So,

$$a^2 + b^2 + 2ab = c^2 + 2ab$$
$$\rightarrow a^2 + b^2 = c^2$$

12. * Given the rectangular prism below, solve for the length of AH.

To solve for AH, we will make a right triangle inside of the rectangular prism with hypotenuse AH and legs AD and DH. Since we know the length of DH (20cm), we just need to solve for the length of AD. Call this length x. Then,

$$
\begin{aligned}
x &= \sqrt{9^2 + 12^2} \\
&= \sqrt{81 + 144} \\
&= \sqrt{225} \\
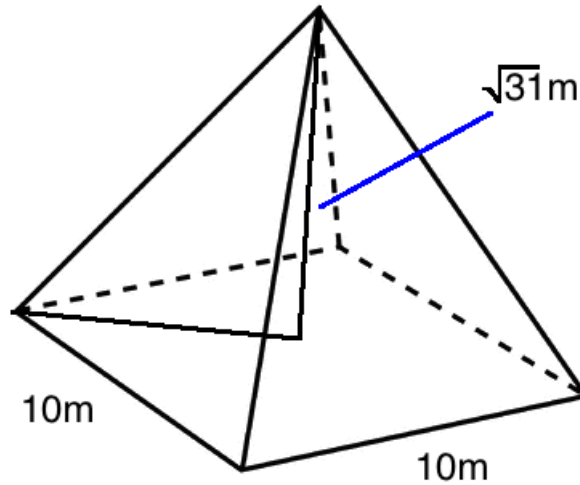&= 15cm
\end{aligned}
$$

Now we can solve for the length of AH. Call this length y. Then,

$$
\begin{aligned}
y &= \sqrt{15^2 + 20^2} \\
&= \sqrt{225 + 400} \\
&= \sqrt{625} \\
&= 25cm
\end{aligned}
$$

Therefore, the length of AH is 25cm.

13. A square-based pyramid has a height of $\sqrt{31}m$ and a base area of $100m^2$. What is the length of the slant of the pyramid (length from the corner of the base to the top of the pyramid)?

Let's draw a picture.

To solve for the length of the slant, we must first solve for the length of the other leg of the right triangle we have drawn inside of the square-based pyramid. The length of this leg is half of the length of the diagonal of the square. Call this length x. Then,
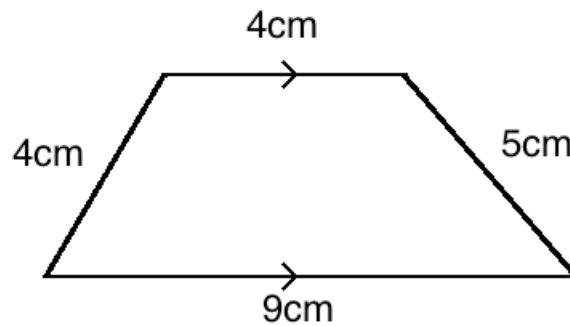
$$
\begin{aligned}
x &= 0.5\sqrt{10^2 + 10^2} \\
&= 0.5\sqrt{200} \\
&= 0.5(10\sqrt{2}) \\
&= 5\sqrt{2}m
\end{aligned}
$$

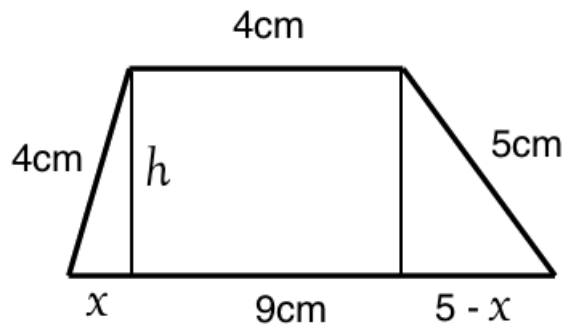Now we can solve for the length of the slant. Call this length s. Then,

$$
\begin{aligned}
s &= \sqrt{(5\sqrt{2})^2 + \sqrt{31}^2} \\
&= \sqrt{25(2) + 31} \\
&= \sqrt{81} \\
&= 9m
\end{aligned}
$$

Therefore, the length of the slant of the pyramid is $9m$.

14. Solve for the area of the trapezoid given the following information (round any calculations to one decimal place).

Let's make some right triangles within the trapezoid:

We cannot find the areas of these two triangles yet, but we can do some algebra to solve for the length of the base of both. We let x be the length of the base of the triangle on the left. This means that the length of the base of the triangle on the right is $5 - x$ ($9 - 5 - x$). We'll now solve for these lengths with the fact that the heights of these two triangles are the same. We have

$$h^2 = 4^2 - x^2$$
$$h^2 = 5^2 - (5 - x)^2$$

So

$$5^2 - (5 - x)^2 = 4^2 - x^2$$
$$\rightarrow \quad 25 - (25 - 10x + x^2) = 16 - x^2$$
$$\rightarrow \quad 25 - 25 + 10x - x^2 = 16 - x^2$$
$$\rightarrow \quad 10x = 16$$
$$\rightarrow \quad x = 1.6m$$

We can now solve for the height of the trapezoid. We'll use the triangle on the right:

$$\begin{aligned} h &= \sqrt{4^2 - 1.6^2} \\ &= \sqrt{16 - 2.56} \\ &= \sqrt{13.44} \\ &\approx 3.7m \end{aligned}$$

Now, for the area of the trapezoid, we'll find the sum of the areas of the two triangles and the rectangle in the middle:

$$\begin{aligned} A &= \frac{1.6 * 3.7}{2} + \frac{3.4 * 3.7}{2} + 4(3.7) & (1) \\ &= \frac{3.7(1.6 + 3.4)}{2} + 4(3.7) & (2) \\ &= \frac{3.7(1.6 + 3.4 + 8(3.7)}{2} & (3) \\ &= 3.7(\frac{5+8}{2}) & (4) \\ &= 3.7(\frac{4+9}{2}) & (5) \\ &= 24.1m^2 & (6) \end{aligned}$$

Notice in line 5 of our work above, we arrived at the formal equation for area of a trapezoid ($A = h(\frac{x+y}{2})$, where x and y are the lengths of the two bases). This is just showing we can find the area in more than one way after we have found the height of the trapezoid.