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Centre for Education in
Mathematics and Computing

# Grade 7 and 8 Math Circles

March 5th/6th/7th

***Inequalities***

## Warm-Up

See how many of the following equations you can solve.

1. $15 \div 3 - 1$
   $= 5 - 1$
   $= 4$

2. $7 \times 3 + 4$
   $= 21 + 4$
   $= 25$

3. $48 \div 4 + 4 \times 5$
   $= 12 + 20$
   $= 32$

4. $48 \div (4 + 4) \times 5$
   $= 48 \div 8 \times 5$
   $= 6 \times 5$
   $= 30$

5. $11 - 6^2 \div 12 + (4 + 5 \times 2) \div 7$
   $= 11 - 6^2 \div 12 + (4 + 10) \div 7$
   $= 11 - 6^2 \div 12 + 14 \div 7$
   $= 11 - 36 \div 12 + 14 \div 7$
   $= 11 - 3 + 2$
   $= 8 + 2$
   $= 10$

6. $3x = 12$
   $\frac{3x}{3} = \frac{12}{3}$
   $x = 4$

7. $2x = -8$
   $\frac{2x}{2} = \frac{-8}{2}$
   $x = -4$

8. $-3x = 6$
   $\frac{-3x}{-3} = \frac{6}{-3}$
   $x = -2$

9. $6x = 18$
   $\frac{6x}{6} = \frac{18}{6}$
   $x = 3$

10. $2x + 5 = 15$
    $2x + 5 - 5 = 15 - 5$
    $2x = 10$
    $\frac{2x}{2} = \frac{10}{2}$
    $x = 5$

11. $3x + 4 = 10$
    $3x + 4 - 4 = 10 - 4$
    $3x = 6$
    $\frac{3x}{3} = \frac{6}{3}$
    $x = 2$

12. $3x + 1 = 2x + 7$
    $3x + 1 - 2x = 2x + 7 - 2x$
    $x + 1 = 7$
    $x + 1 - 1 = 7 - 1$
    $x = 6$

13. $2x + 9 = 6x + 1$
    $2x + 9 - 6x = 6x + 1 - 6x$
    $-4x + 9 = 1$
    $-4x + 9 - 9 = 1 - 9$
    $-4x = -8$
    $\frac{-4x}{-4} = \frac{-8}{-4}$
    $x = 2$

# Review of Order of Operations

When working on more complicated mathematical expressions that involve multiple operations and many steps it is very important to do the math *in the right order* to ensure that you get the right answer.

What is the right order?

1. Begin by simplifying everything inside of **brackets**.

2. Evaluate anything with **exponents**.

3. Starting from the left, work out all of the **multiplication** and **division**, *whichever comes first.*

4. Starting from the left, finish by evaluating all of the **addition** and **subtraction**, *whichever comes first.*

This can be summarized with the acronym **BEDMAS**:

**B**rackets **E**xponents **D**ivision **M**ultiplication **A**ddition **S**ubtraction

**Note:** Remember that division and multiplication are evaluated in the same step, even though D appears before M in the acronym; *division is no more important than multiplication.* Similarly, *addition is no more important than subtraction.*

# Review of Solving Equations

Steps for Solving:

1. Determine what you are trying to isolate/solve for.

2. Simplify the equation as much as possible by adding and subtracting like terms.

   Like terms are terms in a mathematical equation that have the exact same variables; only their coefficients are different.

   You can think of it like adding apples and oranges. If I have 3 apples plus 2 apples plus 5 oranges plus 1 orange, I actually have 5 apples and 6 oranges. Another, more mathematical, example:

   $$5 + x + 3y - 2 - y + 2x = 3 + 3x + 2y$$

3. Isolate the desired variable on one side of the equal sign and everything else on the other side by performing opposite operations *in reverse BEDMAS order*. The goal of isolating a variable, say $x$, is to obtain the form $x = ...$ or $... = x$. Notice that $x$ is positive with a coefficient of 1. There should be no other $x$s on the other side of the equal sign.

| **Original Operation** | | **Opposite Operation** | |
|---|---|---|---|
| Addition | $+$ | $-$ | Subtraction |
| Subtraction | $-$ | $+$ | Addition |
| Division | $\div$ | $\times$ | Multiplication |
| Multiplication | $\times$ | $\div$ | Division |

Reverse BEDMAS (**SAMDEB**):

**S**ubtraction **A**ddition **M**ultiplication **D**ivision **E**xponents **B**rackets

**Note:** Just as before, addition and subtraction have the same priority; as do multiplication and division.

When performing opposite operations, what you do to one side of the equation you **must** do to the other side of the equation.

# Inequalities Definitions

To understand what equations are trying to tell us, we first must understand the symbols being used in those equations.

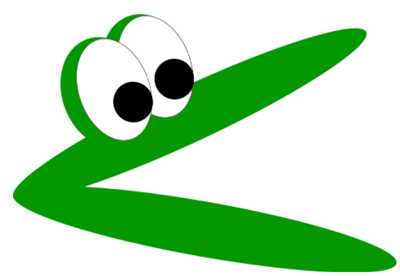
$=$ This symbol means that the first expression is **equal to** the second expression

$<$ This symbol means that the first expression is **less than** the second expression

$>$ This symbol means that the first expression is **greater than** the second expression

In order to know which of these symbols should be used in a specific equation, it can be helpful to rember one of the following sayings:

- The smaller side goes towards the smaller number, the bigger side goes towards the bigger number
- The alligator always wants the bigger meal



**Exercise:**
Determine which symbol belongs in the blank for the following equations to be correct.

1. $5$ ____ $7$
   $<$
2. $7$ ____ $7$
   $=$
3. $7$ ____ $5$
   $>$
4. $3 + 4$ ____ $6 + 1$
   $=$
5. $9 - 5$ ____ $3 + 2$
   $<$
6. $1 + 3$ ____ $12 - 7$
   $<$

# Solving Inequalities

When solving equations with equal signs the expression remains true as long as everything you do to one side, you also do to the other side. When solving inequalities, there are certain operations that are safe to preform and have no effect on the direction of the inequality (which symbol is being used). However, some operations, when preformed on inequalities, cause the direction of the inequality to switch. If the direction of the inequality switches, then $>$ becomes $<$, or $<$ becomes $>$. Below is a list of operations categorized by their effect on the direction of the inequality.

Direction of the Inequality

| Stays the Same | Switches |
|---|---|
| Multiplying both sides by a positive number | Multiplying both sides by a negative number |
| Dividing both sides by a positive number | Dividing both sides by a negative number |
| Adding a number to both sides | Switching left and right sides |
| Subtracting a number from both sides | |
| Simplifying a side | |

**Exercise:**
Determine what the symbol will be after the operation is preformed.

1. $x + 5 > 7$
   Subtracting 5 from both sides
   $>$

2. $3x > 6$
   Dividing both sides by 3
   $>$

3. $x - 5 < 7$
   Adding 5 to both sides
   $<$

4. $12 > -3x$
   Switching left and right sides
   $<$

5. $-3x < 12$
   Dividing both sides by $-3$
   $>$

6. $x + 5 = 7$
   Subtracting 5 from both sides
   $=$
   The equal sign never changes when an operation is applied to both sides.

To solve an inequality you follow the same steps as when solving an equality expression, with the additional step of paying attention to the direction of the inequality. It is normal to have $x$ be on the left side in the final expression. When $x$ is on the left side it is easier to read for meaning as it is of the form $x$ is less than ... or $x$ is greater than ... .

**Exercise:**

Solve the following inequalities for $x$.

1. $x + 5 > 7$

   $x + 5 - 5 > 7 - 5$

   $x > 2$

2. $x - 9 < 7$

   $x - 9 + 9 < 7 + 9$

   $x < 16$

3. $3 < 2 + x$

   $2 + x > 3$

   $2 + x - 2 > 3 - 2$

   $x > 1$

4. $3x > 18$

   $\frac{3x}{3} > \frac{18}{3}$

   $x > 6$

5. $-4x < 16$

   $\frac{-4x}{-4} > \frac{16}{-4}$

   $x > -4$

6. $-3 > -3x$

   $-3x < -3$

   $\frac{-3x}{-3} > \frac{-3}{-3}$

   $x > 1$

7. $1 - x < 7$

   $1 - x - 1 < 7 - 1$

   $-x < 6$

   $\frac{-x}{-1} > \frac{6}{-1}$

   $x > -6$

8. $2x + 5 > 7$

   $2x + 5 - 5 > 7 - 5$

   $2x > 2$

   $\frac{2x}{2} = \frac{2}{2}$

   $x > 1$

# Absolute Values

An **absolute value** is the distance between a number and zero on a number line. This means that all absolute values are positive numbers or zero. Absolute values are represented with the symbols | | surrounding a number. This means that $|-5|$ is the same as $|5|$ as they are both 5 units from zero when observed on a number line, just in different directions.

$$|-5| = 5 = |5|$$

Absolute values play the biggest role when dealing with negative numbers and subtraction. When subtracting two numbers normally, the order in which the numbers are placed matters as it effects whether your answer is positive or negative. Because both the positive and negative numbers have the same absolute value, the order of subtraction does not matter when the absolute value is being considered.

$$|7-5| = |2| = 2$$

$$|5-7| = |-2| = 2$$

When dealing with the multiplication and division by a negative number we see a similar outcome. Normally, whether or not the number is negative has an effect on whether or not the solution is negative, when taking the absolute value, it has no effect. This is again because whether or not the number is negative the absolute value of the number will be the positive equivalent.

$$|2 \times 3| = |6| = 6$$

$$|-2 \times 3| = |-6| = 6$$

$$|2 \times -3| = |-6| = 6$$

If the negative sign is outside the boundaries of the absolute value, then we treat the absolute value as we would a set of brackets, evaluate the inside then apply the outside conditions.

$$-|7-5| = -|2| = -2$$

$$-|5-7| = -|-2| = -2$$

## Solving Absolute Value Equations

When solving an equation that contains an absolute value, we have to account for both the positive and negative values that the absolute value could be. By considering both options, absolute value equations will typically have two correct answers.

**Example:**

$$|2x-1| = 5$$

$$\begin{aligned} 2x - 1 &= 5 \\ 2x &= 6 \\ x &= 3 \end{aligned} \qquad\qquad \begin{aligned} 2x - 1 &= -5 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The same concept applies when there is an inequality. The expression containing the variable must be considered with respect to both the negative and positive value. As we discussed earlier with the direction of inequalities, when the sign has changed on the value we are comparing the expression to, the direction of inequality also has to be switched.

**Example:**

Less Than

$$|2x - 1| < 5$$
$$-5 < 2x - 1 < 5$$
$$-5 + 1 < 2x < 5 + 1$$
$$-4 < 2x < 6$$
$$-2 < x < 3$$

Greater Than

$$|2x - 1| > 5$$
$$2x - 1 > 5 \text{ or } 2x - 1 < -5$$
$$2x > 6 \text{ or } 2x < -4$$
$$x > 3 \text{ or } x < -2$$

**Exercise:**

Solve the following for $x$.

1. $|3x| = 12$

   $3x = 12$ $\frac{3x}{3} = \frac{12}{3}$

   $x = 4$

   or

   $3x = -12$

   $\frac{3x}{3} = \frac{-12}{3}$

   $x = -4$

2. $|2x + 5| = 15$

   $2x + 5 = 15$

   $2x + 5 - 5 = 15 - 5$

   $2x = 10$

   $\frac{2x}{2} = \frac{10}{2}$

   $x = 5$

   or

   $2x + 5 = -15$

   $2x + 5 - 5 = -15 - 5$

   $2x = -20$

   $\frac{2x}{2} = \frac{-20}{2}$

   $x = -10$

3. $|x + 5| > 7$

   $x + 5 > 7$

   $x + 5 - 5 > 7 - 5$

   $x > 2$

   or $x + 5 > -7$

   $x + 5 - 5 > -7 - 5$

   $x < -12$

4. $|x - 9| < 7$

$$-7 < x - 9 < 7$$
$$-7 + 9 < x - 9 + 9 < 7 + 9$$
$$2 < x < 16$$

# Problem Set

1. Determine if the following statements are true or false.

   (a) 0 is less than $-5$
   False

   (b) 7 is less than $-1$
   False

   (c) $-1$ is less than $-3$
   False

   (d) $-8$ is less than $-2$
   True

   (e) $-1$ is less than $|-3|$
   True

   (f) $|5|$ is less than $|7|$
   True

   (g) $|-10|$ is less than $|-9|$
   False

   (h) 0 is less than $|-1|$
   True

2. In a group of five friends:

   - Amy is taller than Carla
   - Dan is shorter than Eric but taller than Bob
   - Eric is shorter than Carla

   Who is the shortest?
   Bob
   Bob $<$ Dan $<$ Eric $<$ Carla $<$ Amy
   2014 Cayley (Grade 10) # 9

3. Five children had dinner. Chris ate more than Max. Brandon ate less than Kayla. Kayla ate less than Max but more than Tanya. Which child ate the second most?
   Max
   2011 Gauss (Grade 8) # 10

4. In downtown Gaussville, there are three buildings with different heights: The Euclid (E), The Newton (N) and The Galileo (G). Only one of these statements below is true.

   (a) The Newton is not the shortest.
   (b) The Euclid is the tallest.
   (c) The Galileo is not the tallest.

   Order the buildings from shortest to tallest in height.
   E, N, G
   2012 Gauss (Grade 8) # 22

5. Solve the following inequalities for $x$.

(a) $3x > 12$

$$\frac{3x}{3} > \frac{12}{3}$$
$$x > 4$$

(b) $2x < -8$

$$\frac{2x}{2} < \frac{-8}{2}$$
$$x < -4$$

(c) $6 < -3x$

$$-3x > 6$$
$$\frac{-3x}{-3} > \frac{6}{-3}$$
$$x < -2$$

(d) $6x < 18$

$$\frac{6x}{6} < \frac{18}{6}$$
$$x < 3$$

(e) $x + 7 < 18$

$$x + 7 - 7 < 18 - 7$$
$$x < 11$$

(f) $x - 1 > 5$

$$x - 1 + 1 > 5 + 1$$
$$x > 6$$

(g) $9 > 6 + x$

$$6 + x < 9$$
$$6 + x - 6 < 9 - 6$$
$$x < 3$$

(h) $2x + 5 > 15$

$$2x + 5 - 5 > 15 - 5$$
$$2x > 10$$
$$\frac{2x}{2} > \frac{10}{2}$$
$$x > 5$$

(i) $3x + 4 < 10$

$$3x + 4 - 4 < 10 - 4$$
$$3x < 6$$
$$\frac{3x}{3} < \frac{6}{3}$$
$$x < 2$$

(j) $3 < 7 - 4x$

$$7 - 4x > 3$$
$$7 - 4x - 7 > 3 - 7$$
$$-4x > -4$$
$$x < 1$$

(k) $3x+1>2x+7$

$$3x+1-2x>2x+7-2x$$
$$x+1>7$$
$$x+1-1>7-1$$
$$x>6$$

(l) $2x+9<6x+1$

$$2x+9-6x<6x+1-6x$$
$$-4x+9<1$$
$$-4x+9-9<1-9$$
$$-4x<-8$$
$$\frac{-4x}{-4}<\frac{-8}{-4}$$
$$x>2$$

6. Solve the following absolute value problems for $x$.

(a) $|3x|=12$

$$3x=12 \text{ or } 3x=-12$$
$$\tfrac{3x}{3}=\tfrac{12}{3} \text{ or } \tfrac{3x}{3}=\tfrac{-12}{3}$$
$$x=4 \text{ or } x=-4$$

(b) $|2x|=8$

$$2x=8 \text{ or } 2x=-8$$
$$\tfrac{2x}{2}=\tfrac{8}{2} \text{ or } \tfrac{2x}{2}=\tfrac{-8}{2}$$
$$x=4 \text{ or } x=-4$$

(c) $6<|3x|$

$$|3x|>6$$
$$3x<-6 \text{ or } 3x>6$$
$$\tfrac{3x}{3}<\tfrac{-6}{3} \text{ or } \tfrac{3x}{3}>\tfrac{6}{3}$$
$$x<-2 \text{ or } x>2$$

(d) $|6x|>18$

$$6x<-18 \text{ or } 6x>18$$
$$\tfrac{6x}{6}<\tfrac{-18}{6} \text{ or } \tfrac{6x}{6}>\tfrac{18}{6}$$
$$x<-3 \text{ or } x>3$$

(e) $|x+7|<18$

$$-18<x+7<18$$
$$-18-7<x+7-7<18-7$$
$$-25<x<11$$

(f) $|x-1|>5$

$$x-1<-5 \text{ or } x-1>5$$
$$x-1+1<-5+1 \text{ or } x-1+1>5+1$$
$$x<-4 \text{ or } x>6$$

(g) $9>|6+x|$

$$|6+x|<9$$
$$-9<6+x<9$$
$$-9-6<6+x-6<9-6$$
$$-15<x<3$$

(h) $|2x+5| > 15$

$$2x+5 < -15 \text{ or } 2x+5 > 15$$
$$2x+5-5 < -15-5 \text{ or}$$
$$2x+5-5 > 15-5$$
$$2x < -20 \text{ or } 2x > 10$$
$$\tfrac{2x}{2} < \tfrac{-20}{2} \text{ or } \tfrac{2x}{2} > \tfrac{10}{2}$$
$$x < -10 \text{ or } x > 5$$

(i) $|3x+4| < 10$

$$-10 < 3x+4 < 10$$
$$-10-4 < 3x+4-4 < 10-4$$
$$-14 < 3x < 6$$
$$\tfrac{-14}{3} < \tfrac{3x}{3} < \tfrac{6}{3}$$
$$\tfrac{-14}{3} < x < 2$$

(j) $3 < |7-4x|$

$$|7-4x| > 3$$
$$7-4x < -3 \text{ or } 7-4x > 3$$
$$7-4x-7 < -3-7 \text{ or}$$
$$7-4x-7 > 3-7$$
$$-4x < -10 \text{ or } -4x > -4$$
$$\tfrac{-4x}{-4} < \tfrac{-10}{-4} \text{ or } \tfrac{-4x}{-4} > \tfrac{-4}{-4}$$
$$x > \tfrac{5}{2} \text{ or } x < 1$$

## CHALLENGE

7. Based on your knowledge of the inequality symbols what do you think is the meaning of $\leq$ and $\geq$?

   $\leq$ means less than or equal to

   $\geq$ means greater than or equal to

8. Solve for $x$.

(a) $3 < \frac{x}{2}$

$$\tfrac{x}{2} > 3$$
$$\tfrac{x}{2} \times 2 > 3 \times 2$$
$$x > 6$$

(b) $4 < \frac{x+6}{3}$

$$\tfrac{x+6}{3} > 4$$
$$\tfrac{x+6}{3} \times 3 > 4 \times 3$$
$$x+6 > 12$$
$$x+6-6 > 12-6$$
$$x > 6$$

(c) $\frac{x}{2} + 9 < 12$

$$\tfrac{x}{2} + 9 - 9 < 12 - 9$$
$$\tfrac{x}{2} < 3$$
$$\tfrac{x}{2} \times 2 < 3 \times 2$$
$$x < 6$$

(d) $-2 < 6x - 2 < 4$

$$-2+2 < 6x-2+2 < 4+2$$
$$0 < 6x < 6$$
$$\tfrac{0}{6} < \tfrac{6x}{6} < \tfrac{6}{6}$$
$$0 < x < 1$$

(e) $2 < \frac{x}{2} + 3 < 5$

$$2 - 3 < \tfrac{x}{2} + 3 - 3 < 5 - 3$$
$$-1 < \tfrac{x}{2} < 2$$
$$-1 \times 2 < \tfrac{x}{2} \times 2 < 2 \times 2$$
$$-2 < x < 4$$

(f) $-2 < \frac{x-7}{3} < 1$

$$-2 \times 3 < \tfrac{x-7}{3} \times 3 < 1 \times 3$$
$$-6 < x - 7 < 3$$
$$-6 + 7 < x - 7 + 7 < 3 + 7$$
$$1 < x < 10$$

(g) $-1 < \frac{2x-5}{3} < 5$

$$-1 \times 3 < \tfrac{2x-5}{3} \times 3 < 5 \times 3$$
$$-3 < 2x - 5 < 15$$
$$-3 + 5 < 2x - 5 + 5 < 15 + 5$$
$$2 < 2x < 20$$
$$\tfrac{2}{2} < \tfrac{2x}{2} < \tfrac{20}{2}$$
$$1 < x < 10$$

(h) $-6 < \frac{3x+6}{2} < 6$

$$-6 \times 2 < \tfrac{3x+6}{2} \times 2 < 6 \times 2$$
$$-12 < 3x + 6 < 12$$
$$-12 - 6 < 3x + 6 - 6 < 12 - 6$$
$$-18 < 3x < 6$$
$$\tfrac{-18}{3} < \tfrac{3x}{3} < \tfrac{6}{3}$$
$$-6 < x < 2$$