Senior Math Circles

Euclidean Geometry
February, 2019

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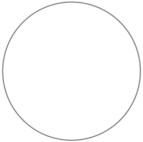
Geometry

This series of sessions is all about Geometry. What do we mean by the term?

- Study of shapes and solids?
- "the branch of mathematics concerned with the properties and relations of points, lines, surfaces, solids, and higher dimensional analogs."
- Geometry vs. Analytic Geometry?
  - Which of these is more accurately called a "circle"?
  - 
  - or "$x^2 + y^2 = 16$"?

Euclid's Elements

In ancient Greece, mathematicians studied geometry extensively, and that work is still used today. One such mathematician was Euclid, who wrote about geometry in his book Elements.

Many truths in geometry which we take for granted are in fact only because Euclid defined them as true, or are a result of Euclid's definitions. Things like:

- The angles in a triangle must add up to 180°.
- The shortest distance between two points is a "straight line".
- Parallel lines never meet.
- A circle has no corners.

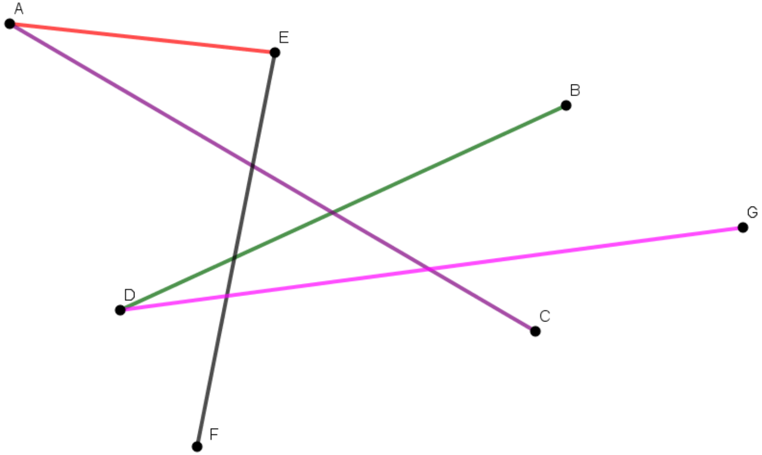
In *Elements*, Euclid begins with 5 postulates, which are basically facts to be taken without proof (also known as axioms).

From these 5 postulates, the rest of what most people think of as geometry is developed. Let's look at the postulates!

Euclid's First Postulate

Postulate 1

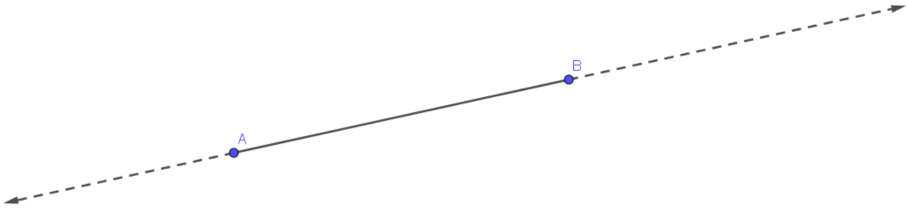
Any two points can be connected with a straight line.



Euclid's Second Postulate

Postulate 2

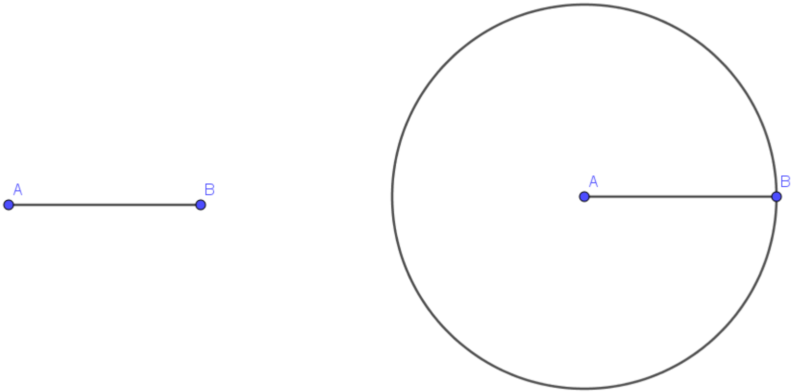
Any straight line segment can be extended indefinitely.



Euclid's Third Postulate

Postulate 3

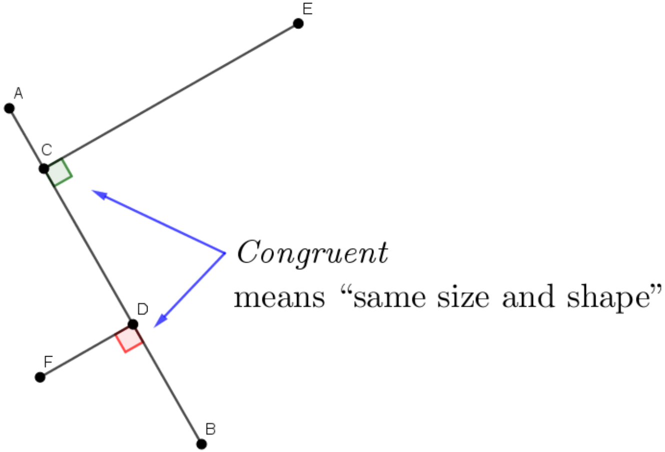
Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.



Euclid's Fourth Postulate

Postulate 4

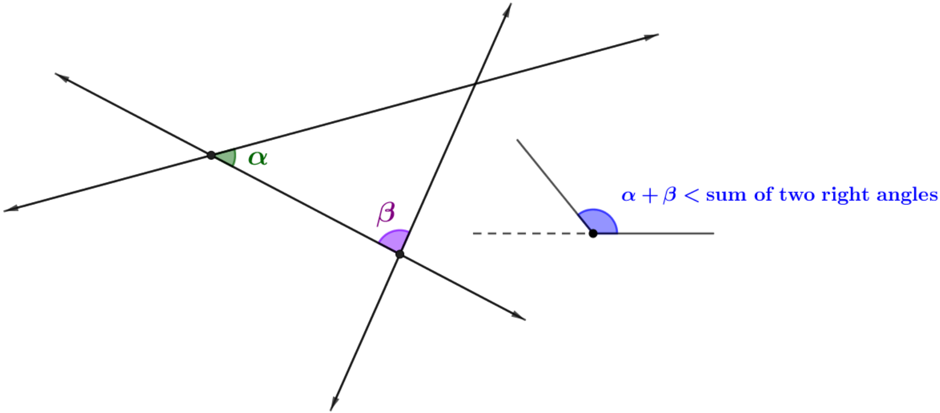
All right angles are congruent.



Euclid's Fifth Postulate

Postulate 5

If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.



Euclid's Fifth Postulate - Comments

- Postulate 5 is stated as a logical implication.
  - That is, it is in the form "If A, then B".
  - We write $A \implies B$.
  - $A =$"two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles"
  - $B =$"the two lines inevitably must intersect each other on that side if extended far enough"
- Any implication can be equivalently stated using what is called its contrapositive.
  - The contrapositive of "If A, then B" is "If not B then not A".
  - We write $\neg B \implies \neg A$.
  - The contrapositive of Postulate 5 is
    "If two lines do not meet each other, no matter how far they are extended on one side of a third line, then they can not be drawn in such a way that the sum of the inner angles each on that side of the third line is less than two right angles."
  - If this contrapositive applies to both sides of the third line then the two lines must never meet.
  - So are they parallel?

Euclid's Fifth Postulate - Comments

- The fifth postulate is also known as the Parallel Postulate, which says
  - Given any straight line and a point not on it, there "exists one and only one" straight line which passes through that point and never intersects the first line, no matter how far they are extended.
- This is how parallel lines are defined.
- It may surprise you to know that Euclid was uncomfortable with this postulate, because it is an implication and it was felt that it should be provable using the first 4 postulates.
  - As it turns out, it is not.
  - So we take it as an axiom.
  - Axioms are statements we take as true without proof.

The Tools of the Ancient Geometers

Given the 5 postulates, all of geometry can be represented using only 3 tools:

- A straight-edge
- A compass
- A pencil

The pencil is so we can actually draw things. Let's look at the other two, and think about what they can do ...

The Straight-Edge

Straight-Edge

Used for drawing straight lines. Nothing more, and nothing less.

- The ancient Greeks did not use rulers! A ruler *is* a type of straight edge but ...
- ... it also measures units.
- A traditional straight-edge has no measurement units and is just used for drawing straight lines.
- Euclid used only the traditional straight-edge.

The Compass

Compass

Used for drawing circles having a given segment as radius and one endpoint as center.

- Compasses are often used to draw multiple circles with equal radii but in truth ...
- ... this relies on the angle of opening at the top of the compass remaining unchanged between the drawing of each circle.
- This is actually practically impossible.
- Compasses that deliberately do not retain the angle are sometimes called collapsing compasses.
- Any compass can be a collapsing compass if you just squeeze it closed when you are finished drawing your circle.
- Euclid used only collapsing compasses.

GeoGebra

https://www.geogebra.org/

GeoGebra is a powerful, free tool that can be used for many mathematical purposes. We will use it to emulate the challenges faced by the ancient Greek geometers. Specifically, we will learn to use 4 basic tools:

The Point Tool
Used to plot a single point anywhere in the plane.

The Line Segment Tool
Used to draw a straight line segment between two points.

The Circle Tool
Used to draw a circle centred at a given point with a given line segment as its radius.

The Intersection Tool
Used to plot a point at the spot where two shapes intersect.

Let's take some time to learn these tools now.

Our Challenges

We will explore Euclidean geometry using only the tools of the ancient Greeks, as well as our ability to think.

Conquering the challenges we'll face generally has three stages:

1. Stating the challenge.
   - What is given?
   - What do we need to construct?
2. Discovering how to do it.
   - The ancient Greeks experimented with compass and straight-edge until they discovered a solution.
   - We will use software.
3. Proving that it is correct.
   - This is *not* the same as doing it a few times where it works.
   - Software doesn't help. Only brains!

Challenge 1: Midpoint of a Line Segment

The Challenge

Given any line segment AB, use a collapsing compass and straightedge to locate the point C on AB such that $AC = CB$.

First, discover the method.

Remember the rules! You may only use

- The line segment tool

- The point tool

- The circle tool

- The intersection tool

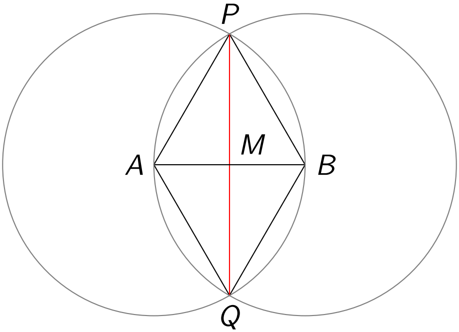
Let's take some time to discover the construction now.

Challenge 1: The Construction

Construction

Draw two circles with equal radius, one with centre A and one with centre B. The radius must be equal to the length of AB. Label the two points of intersection of the circles as P and Q.

Claim: PQ is the perpendicular bisector of AB.



Challenge 1: The Proof, Plus a Bonus Discovery!

Argument	**Reason**
$AP = AQ = BP = BQ$	The four distances are equal to the radius of the circles
$\triangle ABP$ and $\triangle ABQ$ are isosceles	$AP = BP$
$\angle PAB = \angle QAB$	Base angles of isosceles triangles are congruent
$\triangle ABP \cong \triangle ABQ$	Three sides congruent and AB is common to both
$\triangle APQ \cong \triangle BPQ$	Similar reasoning as above
$\triangle AMP$, $\triangle AMQ$, $\triangle BMP$, $\triangle BMQ$ are congruent	Two angles and included side
$AM = BM$	Congruent and isosceles triangles
M is the midpoint of AB	$AM = BM$
BONUS! $PQ \perp AB$	$\angle PMQ = \angle AMQ = \angle QMR = \angle BMP$ and $\angle PMQ + \angle AMQ + \angle QMR + \angle BMP = 360^\circ$

New Tools Unlocked!

Now that we know that we could construct a midpoint of a line segment, and also a perpendicular bisector of a line segment, we have earned the right to use tools designed to do those things for us. So we have unlocked the following:

- The midpoint tool

- The perpendicular bisector tool

Let's take some time to learn these tools now.

Challenge 2: Angle Bisector

The Challenge

Given any three distinct points A, B, and C, construct a point D so that $\angle ABD = \angle CBD$.

Tools:

- The line segment tool

- The circle tool

- The point tool

- The intersection tool

- The midpoint tool

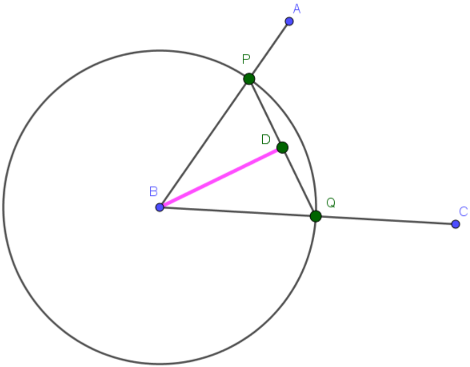
- The perpendicular bisector tool

Let's take some time to discover the construction now.

Challenge 2: The Construction

Construction

Draw circle with centre B, ensuring that the radius is less than both AB and AC.
The circle will intersect line segment AB at a point P and intersect line segment BC at a point Q.
Draw the perpendicular bisector of line segment PQ so that it crosses the point D.
This perpendicular bisector is the angle bisector of $\angle ABC$.



Challenge 2: The Proof

Argument	**Reason**
$BQ = BP$	Radii of the same circle
$PD = DQ$	By construction
$\triangle BPD \cong \triangle BQD$	Corresponding sides equal
$\angle PBD = \angle QBD$	Corresponding angles of congruent triangles are congruent
BD bisects $\angle ABC$	$\angle PBD$ and $\angle QBD$ are adjacent and congruent