Senior Math Circles

Euclidean Geometry
February, 2019

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Today's Plan

Last week we:

- learned about the 5 postulates that serve as the foundation for Euclidean geometry,
- we learned how to do some constructions using only a compass and straightedge (and a pencil),
- and we learned how to prove that the constructions accomplish the task they were designed to accomplish.
- We also earned the "right" to use the midpoint tool and the perpendicular bisector tool in GeoGebra.

Today we will:

- Focus a little more on the proofs,
- earn the right to use a few more of the tools in GeoGebra,
- and investigate some challenging constructions.

Midpoint of a Line Segment, Revisited

Construction

Draw two circles with equal radius, one with centre A and one with centre B. The radius must be equal to the length of AB. Label the two points of intersection of the circles as P and Q.

Claim: PQ is the perpendicular bisector of AB.
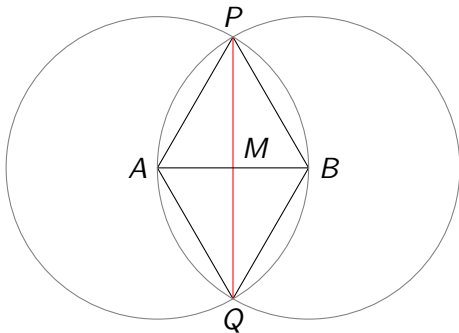

Let's prove this together on the board.

Angle Bisector Revisited

Construction

Draw circle with centre B, ensuring that the radius is less than both AB and AC.
The circle will intersect line segment AB at a point P and intersect line segment BC at a point Q.
Draw the perpendicular bisector of line segment PQ so that it crosses the point D.
Claim: BD is the angle bisector of $\angle ABC$.
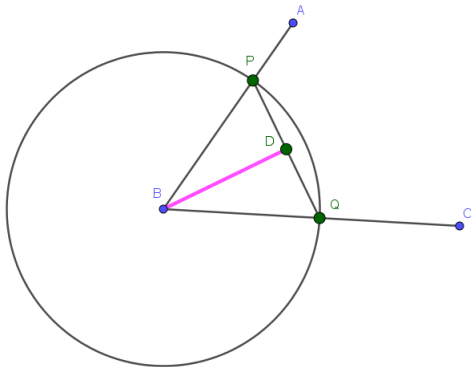
Challenge: Perpendicular and Parallel Lines

The Challenge

Given a line segment AB and a point C not on AB,

a) construct line through C that is perpendicular to the line containing AB.

b) construct line through C that is parallel to the line containing AB.

Tools (may not need all of these):

- The line segment tool

- The circle tool

- The point tool

- The intersection tool

- The midpoint tool

- The perpendicular bisector tool
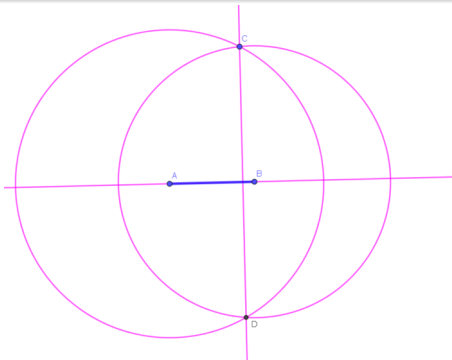
- The angle bisector tool

Let's take some time to discover the constructions now.

Perpendicular and Parallel Lines: Construction

Perpendicular Line Construction

1. Draw circle centred at A through C
2. Draw circle centred at B through C
3. Label the other point of intersection of these two circles D

Claim: The line through DC is perpendicular to the line through AB.
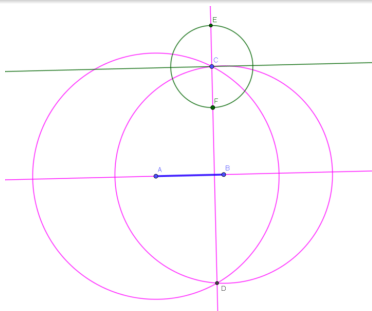
Perpendicular and Parallel Lines: Construction

Parallel Line Construction

1. Construct CD as above
2. Draw any circle centred at C
3. Label E and F as the points of intersection with this circle and the line CD extended.
4. Construct the perpendicular bisector of EF

Claim: This perpendicular bisector passes through C and is parallel to AB.

Pythagorean Theorem

Before we try the next few challenges, let's have some fun and prove the Pythagorean Theorem. There are actually many ways to do this, but we will use the method Euclid used. Consider this diagram:
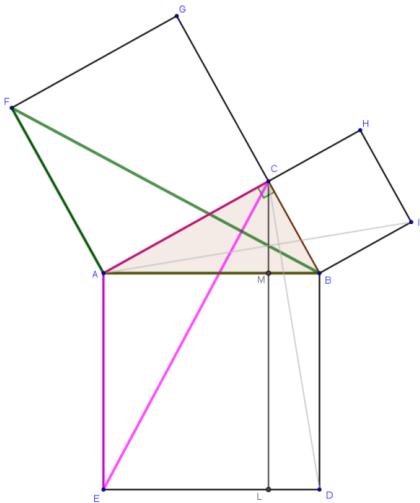
Let's continue on the board!

More Challenges!

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10. **Square by Adjacent Midpoints:** Give two points A and B, construct a square $CDEF$ so that A is the midpoint of CD and B is the midpoint of CF.