Combinatorial Order and Chaos, Week 1

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"You can always find order within chaos."

1 Class Summary

The Pigeonhole Principle states that if we must place $n+1$ pigeons in n pigeonholes, there must be at least two pigeons that are placed in the same hole.

The Generalised Pigeonhole Principle states that if we must place at least $kn+1$ pigeons in n pigeonholes, there must be at least $k+1$ pigeons that are placed in the same hole.

Brain Teaser 1.1. Prove the Generalised Pigeonhole Principle.

Proof. We can prove this by contradiction: Suppose there are at most k pigeons in any given pigeonhole. Then there are at most kn pigeons distributed across the n pigeonholes, contradicting the fact that we have placed at least $kn+1$ pigeons in these pigeonholes. □

Here are some basic (but amusing) applications of the Pigeonhole Principle:

- There are two non-bald people living in the Kitchener-Waterloo region who have the same number of hairs on their head.
- If you wake up one morning not wanting to turn your light on and you only have plain black, white, purple, and yellow socks, you only need to grab five socks to make a matching pair.
- For every 27 consecutive words in this handout, two words must start with the same letter.

Brain Teaser 1.2. Prove that there are two people in the Math Circles group who know the same number of people in the group.

Proof. We could go around and ask everyone in the room how many other people here they know, but it's much easier to approach this in general. Let's ask ourselves what would happen if this wasn't true. If there are n people in this room, that means that there's one person who knows 0 other people, one person who knows 1 other person, one person who knows 2 other people, and so on, until we get one person who knows all of the $n-1$ other people in this room.

But wait! We can't have one person who knows nobody else in the room and another person who knows everybody else in the room! So it's impossible for everybody in the room to know a different number of people, and thus by the Pigeonhole Principle, there are two different people in the room who know the same number of people. □

Brain Teaser 1.3. Suppose I pick five points in a square of area 4. Prove that there must be two points separated by a distance of at most $\sqrt{2}$.

Proof. If we could divide up our square into four smaller regions, we'd know that two of our points have to be in the same region. There's a nice way we could do this: Divide the square in half lengthwise and widthwise to get four squares of area 1. By the Pigeonhole Principle, two points must be in the same square. The farthest that two points could be away from each other is at two opposite corners of the square, so they must thus be separated by a distance of at most $\sqrt{2}$! □

Brain Teaser 1.4. Prove that:

(a) For any set of 21 positive integers, there is some subset of 5 of them whose sum is a multiple of 5.

(b) For any set of 17 positive integers, there is some subset of 5 of them whose sum is a multiple of 5.

Proof. We can start by saying that any integer divided by 5 must leave a remainder of 0, 1, 2, 3, or 4. With this in mind, we can apply the Pigeonhole Principle to see that when we have 21 positive integers, there must be five of them with the same remainder. When we add them up, we'll get a number that's a multiple of 5.

But just applying the Pigeonhole Principle to part (b) doesn't seem to give us much - it only tells us that four numbers have to have the same remainder. If we could get five numbers with the same remainder instead, that'd be nice, since we could follow our reasoning from part (a) and add those five numbers with the same remainder up to get a number that's divisible by 5. And indeed, we could do that if we only had four remainders to worry about. In other words, when we divide our 17 numbers by 5 and look at the remainders, if we only get four or fewer distinct remainders, then we can use the Pigeonhole Principle and we're done.

But what about the case where we divide our 17 numbers by 5 and all five remainders are present? In that case, we can take one number that leaves each distinct remainder and add them up. Then we'll have a number that's divisible by 5, and we're done. □

Brain Teaser 1.5. Given any set of ten distinct 2-digit numbers, prove that there exist two disjoint subsets (of the 10 numbers) with the same sum.

Proof. There's a lot of possibilities here. We could experiment with a fixed set of ten 2-digit numbers, but that doesn't help us show that this is true for every such set. We could try and figure out some things that are true about every set, then: For example, we know that the smallest possible number in a set is 10 and the largest possible number in a set is 99. Now, if we want two disjoint subsets of these ten numbers that have the same sum, then the empty set won't be one of these subsets, and the set of all ten numbers won't be either. Thus the smallest possible subset sum would be 10 and the largest possible subset sum would be $91 + 92 + \cdots + 99 = 855$. We don't really know if every subset sum in the middle can be attained, but that means there's at most 846 distinct subset sums that could be attained.

On the other hand, let's see how many subsets of the 10 numbers there are. Each of the numbers could either be in a subset or not be in a subset, so there are $2^{10} = 1024$ subsets of those ten numbers, and we don't want to count the empty set or the set of all ten numbers, so there's really only 1022 subsets we care about. The Pigeonhole Principle now tells us that two of our subsets have the same sum!

But we're not quite done yet. If our subsets are disjoint, then we're done. But what if the subsets aren't disjoint? That means they have some numbers in common, which contribute to both subset sums. Why not remove those numbers from both sets? That way we'll still get subsets with the same sum, and those subsets will be disjoint - so we're done! □

Brain Teaser 1.6. If n is an integer not divisible by 2 or 5, prove that n has a multiple consisting entirely of "1" digits.

Proof. We could test a few examples, but we'll quickly find that our numbers get too big too fast - we have that $3 \cdot 37 = 111$, $7 \cdot 15873 = 111111$, and $9 \cdot 12345679 = 111111111$. Yikes!

Instead, let's try and follow the same strategy we used earlier when we were looking at multiples of 5. Let's define a sequence a_n of numbers so that a_i is the number consisting of i ones - so $a_1 = 1, a_2 = 11, a_3 = 111$, and so on. If we look at enough terms in our sequence ($n+1$, to be precise), the Pigeonhole Principle will tell us that we'll stumble upon two terms $a_j < a_k$ so that a_j and a_k leave the same remainder when divided by n. If that remainder is zero, then we're done and both of these are multiples of n consisting entirely of "1" digits. But if it's not, it looks like we're stuck...

Or are we? If a_j and a_k have the same remainder when divided by n, that means that $a_k - a_j$ will be a multiple of n. This number consists of a bunch of ones followed by a bunch of zeroes. To be precise, it consists of $k - j$ ones followed by j zeroes, so $a_k - a_j = a_{k-j} \cdot 10^j$. But n isn't divisible by 2 or 5, which means that a_{k-j} is divisible by n (since 10^j can't be)! □

2 Brain Teasers

Warming Up the Grill

Brain Teaser 2.1. Everyone in a group of people is asked to name their favourite letter of the English alphabet, their favourite integer between 1 and 10 inclusive, and their birthday. How large does the group need to be to ensure that there are at five people who give identical responses?

Brain Teaser 2.2. Prove that given any 10 integers, either one of them is divisible by 10 or there are two numbers that differ by a multiple of 10.

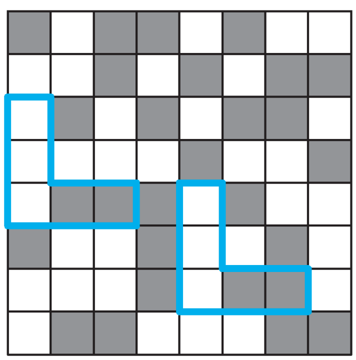
Brain Teaser 2.3. Prove that among any 11 positive integers from the set $\{1, 2, \cdots, 20\}$, there must be two of them that sum to 21.

Brain Teaser 2.4. Prove that if we place ten points in a regular triangle of side length 3, we can find two points that are at most a distance of 1 apart from each other.

Brain Teaser 2.5. I label the 12 points of a dodecagon with the numbers 1, 2, ..., 11, 12. Prove that I can find three consecutive vertices of the dodecagon whose labels sum up to at least 20.

Searing the Steak

Brain Teaser 2.6. The squares of an 8×8 grid are coloured black or white. Let's use the term *L-region* for 5 squares arranged in an L, as shown in the picture (note that the corner of the L must be in its lower left). Prove that no matter how we colour the grid, there must be two distinct L-regions (partial overlap allowed) that are coloured identically.



Brain Teaser 2.7. A lattice point is a point with integer coordinates. Nine lattice points are chosen in 3D space. Prove that there must be some two points you chose whose midpoint is also a lattice point.

Brain Teaser 2.8. Let S be a 51-element subset of $\{1, 2, \cdots, 100\}$. Prove that:

(a) I can find two numbers in S which are relatively prime.

(b) I can find two numbers in S such that one divides the other.

Brain Teaser 2.9. I play at least one game of chess every day for eight weeks, but no more than 11 games in any 7-day period. Show that there is some period of consecutive days when I play exactly 23 games.

Brain Teaser 2.10. Suppose I have some family $\mathcal{F}$ of subsets of $\{1, 2, 3, \cdots, n\}$, so that for sets $X, Y \in \mathcal{F}$, $X \cap Y \neq \emptyset$. Prove that $|\mathcal{F}| \leq 2^{n-1}$, and find such a family $\mathcal{F}$ where $|\mathcal{F}| = 2^{n-1}$.

Brain Teaser 2.11. Prove that for every prime p except 2 and 5, there is a power of p that ends with the digits 0001.

Dinnertime

Brain Teaser 2.12. Suppose we are given five points on a sphere.

(a) It is easy to see that there must be a closed hemisphere containing three of these points - why is this the case? (A closed hemisphere is half a sphere, including its boundary.)

(b) Must there exist a closed hemisphere containing four of these points?

Brain Teaser 2.13. Suppose I take a set of 100 integers (not necessarily distinct). Prove that I can find some subset of these integers that have a sum divisible by 100.

Brain Teaser 2.14. Suppose we place ten points in a circle of diameter 5. Prove that there exist two points that are a distance of at most 2 from each other.

Brain Teaser 2.15. The Fibonacci sequence is defined by letting $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Show that there exists a F_n that is divisible by 10^{2019}.

Brain Teaser 2.16. Every point in the plane is coloured red, blue, or green. Prove that I can find some rectangle where all the points are coloured with the same colour.

Brain Teaser 2.17. Suppose $mn + 1$ people of different heights line up. Prove that either we can pick out $m + 1$ people (not necessarily consecutive) who are arranged in increasing order by height, or $n+1$ people (not necessarily consecutive) who are arranged in decreasing order by height.

Solutions to these problems will not be given. For hints or solutions to individual problems, feel free to email the presenter at kris.siy@uwaterloo.ca.