Math Circles. Group Theory. Solution Set 1.

Diana Carolina Castañeda Santos
dccastan@uwaterloo.ca
University of Waterloo

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1 Solutions

1. Let $(2\mathbb{Z}, +)$ denote the set of even integers $\{\cdots, -4, -2, 0, 2, 4, 6, \cdots\}$ with the usual addition.

   (a) What is the identity in this group?
   The identity is 0.

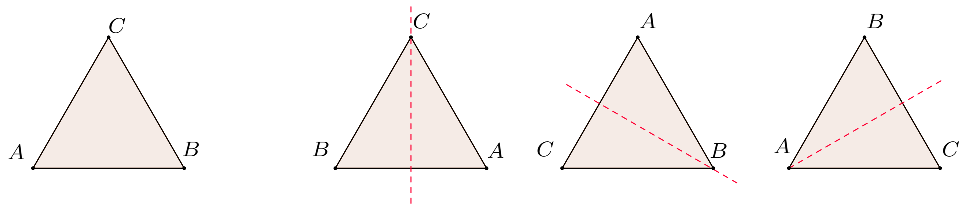
   (b) What is the inverse of 2?
   The inverse of 2 is -2.

   (c) What is the inverse of -100?
   The inverse is 100.

   (d) What is the inverse of an element a in $2\mathbb{Z}$?
   The inverse of a is $-a$.

   (e) What is the relationship between $(2\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$?
   Notice that $(2\mathbb{Z}, +)$ is a group with the usual operation of addition in $\mathbb{Z}$, and $2\mathbb{Z} \subset \mathbb{Z}$. Since they are preserving the same operation and one is contained in the other, we say that $(2\mathbb{Z}, +)$ is a *subgroup* of $(\mathbb{Z}, +)$.

2. Let's find the Dihedral group D_3, the group of symmetries of an equilateral triangle.Let e be the fixed rotation (or $0°$ rotation), R be the $60°$ clockwise rotation, R^2 be the $120°$ clockwise rotation. To name the flips, let's use the following notation:



The first triangle is the triangle in the initial position. We call V the flip made in the second figure. Similarly we call D the flip of the third figure, and D' the flip of the last figure.

Complete the operation table for D_3.

$\cdot$	e	R	R^2	V	D	D'
e	e	R	R^2	V	D	D'
R	R	R^2	e	D'	V	D
R^2	R^2	e	R	D	D'	V
V	V	D	D'	e	R	R^2
D	D	D'	V	R^2	e	R
D'	D'	V	D	R	R^2	e

3. Complete the multiplication table of $(\mathbb{Z}_7, +)$.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

4. Can we think on multiplication in $\mathbb{Z}_5$? Complete the multiplication table for $(\mathbb{Z}_5, \cdot)$ and figure out if it satisfies the nice properties of having an identity element and inverses. If not, can you fix it?

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Notice that if we exclude 0, we obtain a group. 1 is the identity element and every element has an inverse. $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, and $4^{-1} = 4$. Hence $(\mathbb{Z}_5 \setminus \{0\}, \cdot)$ is a group (You can verify that it is associative).

5. Find all values of x in $\mathbb{Z}_{10}$ that satisfy the equation $3x + 9 = 1 \pmod{10}$.

$$\begin{aligned} 3x + 9 &= 1 \pmod{10} \\ 3x &= -8 \pmod{10} \\ 3x &= 2 \pmod{10} \end{aligned}$$

We try all values of x from 0 to 9 and we see that $x = 4$ is the only solution to this equation.

6. Does the equation $x^2 = -1 \pmod 5$ have solutions in $\mathbb{Z}_5$?

We can replace x by the possible values that it could be in $\mathbb{Z}_5$.

x	0	1	2	3	4
x^2	0	1	4	4	1

Recall that $-1 = 4 \pmod 5$. Hence $x = 2$ and $x = 3$ are the solutions of the equation.

7. The set of natural numbers $\mathbb{N}$ with the usual addition doesn't form a group. Can you guess why this is not a group?

Natural numbers don't form a group because they don't satisfy the property of the inverses. For example, $3 \in \mathbb{N}$ and there is no number $a \in \mathbb{N}$ such that

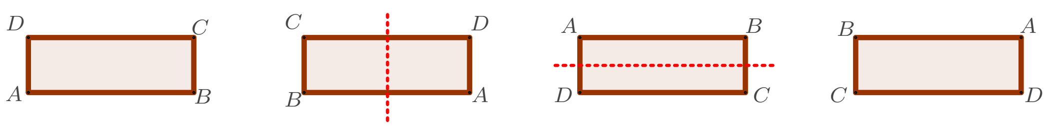
$3 + a = a + 3 = 0$.

8. Find elements A, B, C in D_4 such that $AC = CB$, but $A \neq B$. This shows that the "cross cancellation" is not valid!

   For example $R \cdot H = H \cdot R^3$, but we can't cancel H since $R \neq R^3$. This is an important fact of groups that are not abelian. The order of operations matter and in some cases the cross cancellation is not permitted.

9. Describe the symmetries of a non-square rectangle. Draw out the multiplication table of this group.

   This group has the following symmetries:



   The first represents the fixed move which we denote by I, the second represents the vertical flip which we denote by V, the third represents the horizontal flip which we denote by H, and the last one represents the rotation by $180°$ which we denote by R. The multiplication table for these moves is:

$\cdot$	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I