Combinatorial Order and Chaos, Week 2

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"You can always find order within chaos."

1 Class Summary

Brain Teaser 1.1. I go to a party with six people (including me). Prove that there's either three people there who are mutual acquaintances, or three people there who are mutual strangers.

Proof. Let's start by looking from my perspective. There's five other people at the party, so no matter what kind of party I'm walking into, I'll either be acquaintances with three people or strangers to three people. Let's suppose I know three people (the other case will be similar). If any two of them also know each other, then we've got three people who are mutual acquaintances. If that's not true, then no three of them know each other, and we've got three people who are mutual strangers! □

Note that this will not necessarily be true if there are only five people at the party - why is this the case?

Brain Teaser 1.2. I go to a party with a lot of people. Prove that if our party is big enough, then there's either n people there who are mutual acquaintances, or n people there who are mutual strangers.

Proof. We can try the same thing as before. Let's pick a person and look from their perspective. Call them person 1. They'll either be acquainted with more people in the room, or they'll be strangers to more people in the room. (If they're somehow equal, we can pick one.)

Now, assume they know a lot of people in the room (the other case will be similar), and consider the group of people that they know. We can do exactly the same thing: Pick a person 2, and they'll either be acquainted with more people in that group or be strangers to more people in that group. Let's suppose this time that they're strangers to more people. By considering the group of people they're strangers to and picking a person 3 in that group, we can repeat this process again, and again, and again, since we assumed our party is big enough.

Eventually, we can take all the people we've picked and line them up in order, so that person 1 is first, person 2 is on their right, person 3 is on person 2's right, and so on. We know a few things about all of these people: Each person was picked in the group of people that the previous person is acquainted with (if they were acquainted with more people) or the group of people that the previous person was strangers to (if they were strangers to more people). We narrowed the group in selecting people each time, so each successive person must also be in that group. That means that if we pick any person, they'll either be acquainted with all the people to their right or be strangers to all the people on their right!

Finally, let's label all of the people in our line: Call each person a "socialite" if they're acquainted with everyone to their right, and a "hermit" if they're strangers to all the people on their right. If we have $2n-1$ people in our line, by the Pigeonhole Principle, this means that there must be either n socialites or n hermits. So we know that there are either n people who are mutual acquaintances or n people who are mutual strangers!

We can also check how big our party needs to be for this to actually happen: If we have a party with $4^{n-1} = 2^{2n-2}$ people, then by the Pigeonhole Principle, the group that person 2 is in must have size at least 2^{2n-3}, the group that person 3 is in must have size 2^{2n-4}, and so on, so that our final person, person $2n-1$, is picked from a group of size $2^0 = 1$. $\square$

In general, we use the notation $R(m, n)$ to denote the minimum number of people that we need to have at a party to guarantee that either m people there are mutual acquaintances, or n people there are mutual strangers. For example, we've just shown that $R(3, 3) = 6$ and that $R(n, n) \leq 4^{n-1}$ for all n. These are called *Ramsey numbers*.

While we know that Ramsey numbers exist and we know the exact value of small Ramsey numbers, we actually know very little about them in general. In fact, the bound $R(n, n) \leq 4^{n-1}$ is just about the best general upper bound we have for them at the moment, and that's not a very good bound - if the whole world went to a party, we'd only know that there were either 16 mutual acquaintances or 16 mutual strangers (and common sense that it's most likely the latter). On finding the exact values of Ramsey numbers, the late great mathematician Paul Erdős, who was the first to show that $R(n, n) \leq 4^{n-1}$ once said:

> Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number $R(5, 5)$. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number $R(6, 6)$, however, we would have no choice but to launch a preemptive attack.

(For reference, the bounds we have at the moment for these Ramsey numbers are $43 \leq R(5, 5) \leq 48$ and $102 \leq R(6, 6) \leq 165$. If you'd like more information on what we know about other small Ramsey numbers at the moment, Wikipedia has a pretty good reference.)

There are many generalisations of Ramsey's theorem which can be used to obtain beautiful results. One of these is as follows: Suppose that at a party, every possible group of k people gets to talk with each other, and these groups were either loud or quiet. Then if we have enough people at the party, there will either be m people where every subset of k people talked loudly, or n people where every subset of k people talked quietly. You'll prove this and other generalisations in later exercises, but we'll take a look at one particular geometric application of this generalisation.

A set of points in the plane is said to be in *general position* if no three of them are collinear, and the *convex hull* of a finite set of points in general position is the smallest convex polygon that contains all of the points. You can imagine that you're stretching a rubber band so that it contains the entire set of points; when you release it, you'll get the convex hull of these points.

Brain Teaser 1.3. Show that for any five points in the plane in general position, some four of them are the vertices of a convex quadrilateral.

Proof. Look at the convex hull of the five points. By definition, any point not on the convex hull will be inside the hull. Of course, if the convex hull is a pentagon or quadrilateral, then we're done, since considering any four points in the convex hull will give us a convex quadrilateral. But what if it's a triangle? Then there will be two points inside the triangle. Those two points will lie on a unique line, which divide our triangle into two. But now, the half of the triangle that contains two other points along with our two interior points will now form a convex quadrilateral! □

Brain Teaser 1.4. Show that if n points are in general position and every quadrilateral formed from these n points is convex, then the n points are the vertices of a convex n-gon.

Proof. Let's suppose that the n points aren't the vertices of a convex n-gon and go for a contradiction. The convex hull of these n points must contain some interior point P. If we take a vertex Q of the convex hull and draw line segments from it to every other vertex of the hull, we'll divide the hull into triangles, and P will end up inside one of the triangles. But now the three vertices of this triangle along with P won't form a convex quadrilateral, a contradiction. □

Brain Teaser 1.5. Show that for every integer $m \geq 3$, there exists an integer n so that whenever n points in the plane are in general position, some m of these points are the vertices of a convex m-gon.

Proof. Let's pick n to be the number of people needed at a party so that every possible group of 4 people talks with each other, and either there are 5 people where every subset of 4 people talks loudly or m people where every subset of 4 people talks quietly.

Now, suppose that we have n points in general position. For every subset of 4 of these points, the convex hull of these 4 points will either be a triangle or a quadrilateral. Let's label these subsets of 4 points: If the convex hull is a triangle, let's label it "loud", and if it's a quadrilateral, we'll label it "quiet". Our theorem now tells us that there's either 5 points where every subset of 4 points is loud, or m points where the every subset of 4 points is quiet. But the first case isn't possible, since we know that among five points, four of those points must form a convex quadrilateral! So there must be m points where every subset of 4 points forms a convex quadrilateral, and so these m points are the vertices of a convex m-gon. □

This problem was coined the "Happy Ending Problem" because its solution led to the marriage of two mathematicians, George Szekeres and Esther Klein. You might also be curious as to just how many points we need in general position to find a convex m-gon. For $m = 3$, we need 3 points. We showed that for $m = 4$, we need 5 points. It turns out that for $m = 5$, we need 9 points, and for $m = 6$, we need 17 points. We don't have any exact values for $m > 6$ yet, but Erdős and Szekeres conjectured in their original paper where they proved this that in general, at least $1 + 2^{m-2}$ points are needed, and later showed through constructing examples that at least that many points are needed.

Notice that we have a common theme with everything we've done. When we looked at the Pigeonhole Principle, we saw many examples where we could show that some "pigeonhole" had a given number of "pigeons", but there was no way to actually go about explicitly finding this pigeonhole. With the party problem, we knew that there existed a set of mutual acquaintances or mutual strangers but didn't necessarily find that set of people; now, we know that there's a set of points forming a convex m-gon, but we don't necessarily know how to find it! These are famous examples in a branch of mathematics called *Ramsey theory*, which studies the relation between order and chaos. The main idea of Ramsey theory is that as long as we take a structure that's big enough, some kind of order is sure to exist within it, no matter how chaotic it is. Hopefully, the last few weeks have shown you that this is a valuable mathematical idea!

2 Brain Teasers

Lighting a Singing Candle

Brain Teaser 2.1. I go to a party with 17 people. Any two people at the party are either friends, enemies, or strangers. Prove that there are either three people at the party who are mutual friends, three people at the party who are mutual enemies, or three people at the party who are mutual strangers.

Brain Teaser 2.2. Find, with justification, the Ramsey numbers $R(1, n)$ and $R(2, n)$ for all positive integers n.

Brain Teaser 2.3. Explain why $R(m, n) = R(n, m)$ for all positive integers m and n.

Brain Teaser 2.4. For this Brain Teaser, you can assume that $R(3, 4) = 9$.

(a) Show that $R(4, 4) \leq 18$.

(b) Show that $R(3, 5) \leq 14$.

Brain Teaser 2.5. Find a good way of explaining to one of your classmates why finding the exact value of big Ramsey numbers is really, really hard.

The Singalong

Brain Teaser 2.6. Show that $R(3, 4) = 9$.

Brain Teaser 2.7.

(a) Show that $R(m,n) \leq R(m-1,n) + R(m,n-1)$ for all positive integers $m, n > 2$.

(b) Hence show that $R(m,n) \leq \binom{m+n-2}{m-1}$ for all positive integers.

Brain Teaser 2.8. Another famous question in geometrical Ramsey theory is that of the *chromatic number of the plane*: Let's try and colour the plane with n different colours. What's the minimum number of colours we need to ensure that no two points a distance of exactly 1 apart are the same colour?

(a) Suppose we colour every point in the plane red or blue. Show that we can always find two points of the same colour a distance of exactly 1 apart.

(b) Suppose we colour every point in the plane red, blue, or green. Show that we can always find two points of the same colour a distance of exactly 1 apart.

(c) Show that it is possible to colour every point in the plane with one of nine colours so that no two points of the same colour will be a distance of exactly 1 apart.

You've just shown that the chromatic number of the plane is greater than 3, but less than or equal to 9. At the moment, we know that the chromatic number of the plane is either 5, 6, or 7, but we don't know its exact value.

Help, the Candle Won't Stop Singing

Brain Teaser 2.9. Show that in a party with six people, there must in fact be *two* groups of three people who are either mutual acquaintances or mutual strangers. (These groups can overlap, and it's also possible to have one group of mutual acquaintances and one group of mutual strangers.)

Brain Teaser 2.10.

(a) I go to a party with an infinite number of people. Prove that there's either an infinite number of people there who are mutual acquaintances, or an infinite number of people there who are mutual strangers.

(b) Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of real numbers. Prove that this sequence contains an infinite monotone subsequence: That is, there exists an infinite sequence of positive integers $x_1 < x_2 < x_3 < \cdots$ such that either $a_{x_1} < a_{x_2} < a_{x_3} < \cdots$ or $a_{x_1} > ax_2 > a_{x_3} > \cdots$.

Brain Teaser 2.11.

(a) I went to a party with t^{kt} people. Everyone at the party talked with everyone else at the party. They all talked about t different topics (where t is a positive integer), and each pair of people at the party talked about one topic. Show that there are k people who all talked about the same topic among themselves.

(b) Show that for each positive integer t, there exists a positive integer n such that if we partition the set $\{1, 2, \cdots, n\}$ into subsets $(A_1, A_2, \cdots, A_t)$, then there exists some A_i and some integers $x, y, z \in A_i$ (not necessarily distinct) such that $x + y = z$. (This is known as Schur's Theorem.)

Brain Teaser 2.12.

(a) Find a set of eight points in the plane in general position so that no five of them are the vertices of a convex pentagon.

(b) Show that for any nine points in the plane in general position, some five of them are the vertices of a convex pentagon.

Tossing the Candle Out the Window

Brain Teaser 2.13.

(a) Show that $R(3, 5) = 14$.

(b) Show that $R(4, 4) = 18$.

Brain Teaser 2.14. Prove the generalisation of Ramsey's theorem we used to prove the Happy Ending Theorem: Suppose that at a party, every possible group of k people gets to talk with each other, and these groups were either loud or quiet. Then if we have enough people at the party, there will either be m people where every subset of k people talked loudly, or n people where every subset of k people talked quietly.

Brain Teaser 2.15. Show that $R(k, k) \geq 2^{\frac{k}{2}}$ for all positive integers $k \geq 3$. (This is basically the best general lower bound for Ramsey numbers that we have at the moment, and was also proven by Erdős.)

Solutions to these problems will not be given. For hints or solutions to individual problems, feel free to email the presenter at kris.siy@uwaterloo.ca.