Combinatorial Order and Chaos, Week 2

March 6, 2019

Lighting a Singing Candle

Brain Teaser 1. I go to a party with 17 people. Any two people at the party are either friends, enemies, or strangers. Prove that there are either three people at the party who are mutual friends, three people at the party who are mutual enemies, or three people at the party who are mutual strangers.

Brain Teaser 2. Find, with justification, the Ramsey numbers $R(1, n)$ and $R(2, n)$ for all positive integers n.

Brain Teaser 3. Explain why $R(m, n) = R(n, m)$ for all positive integers m and n.

Brain Teaser 4. For this Brain Teaser, you can assume that $R(3, 4) = 9$.

(a) Show that $R(4, 4) \leq 18$.

(b) Show that $R(3, 5) \leq 14$.

Brain Teaser 5. Find a good way of explaining to one of your classmates why finding the exact value of big Ramsey numbers is really, really hard.

The Singalong

Brain Teaser 6. Show that $R(3, 4) = 9$.

Brain Teaser 7.

(a) Show that $R(m, n) \leq R(m-1, n) + R(m, n-1)$ for all positive integers $m, n > 2$.

(b) Hence show that $R(m, n) \leq \binom{m+n-2}{m-1}$ for all positive integers.

Brain Teaser 8. Another famous question in geometrical Ramsey theory is that of the *chromatic number of the plane*: Let's try and colour the plane with n different colours. What's the minimum number of colours we need to ensure that no two points a distance of exactly 1 apart are the same colour?

(a) Suppose we colour every point in the plane red or blue. Show that we can always find two points of the same colour a distance of exactly 1 apart.

(b) Suppose we colour every point in the plane red, blue, or green. Show that we can always find two points of the same colour a distance of exactly 1 apart.

(c) Show that it is possible to colour every point in the plane with one of nine colours so that no two points of the same colour will be a distance of exactly 1 apart.

You've just shown that the chromatic number of the plane is greater than 3, but less than or equal to 9. At the moment, we know that the chromatic number of the plane is either 5, 6, or 7, but we don't know its exact value.

Help, the Candle Won't Stop Singing

Brain Teaser 9. Show that in a party with six people, there must in fact be *two* groups of three people who are either mutual acquaintances or mutual strangers. (These groups can overlap, and it's also possible to have one group of mutual acquaintances and one group of mutual strangers.)

Brain Teaser 10.

(a) I go to a party with an infinite number of people. Prove that there's either an infinite number of people there who are mutual acquaintances, or an infinite number of people there who are mutual strangers.

(b) Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of real numbers. Prove that this sequence contains an infinite monotone subsequence: That is, there exists an infinite sequence of positive integers $x_1 < x_2 < x_3 < \cdots$ such that either $a_{x_1} < a_{x_2} < a_{x_3} < \cdots$ or $a_{x_1} > ax_2 > a_{x_3} > \cdots$.

Brain Teaser 11.

(a) I went to a party with t^{kt} people. Everyone at the party talked with everyone else at the party. They all talked about t different topics (where t is a positive integer), and each pair of people at the party talked about one topic. Show that there are k people who all talked about the same topic among themselves.

(b) Show that for each positive integer t, there exists a positive integer n such that if we partition the set $\{1, 2, \cdots, n\}$ into subsets $(A_1, A_2, \cdots, A_t)$, then there exists some A_i and some integers $x, y, z \in A_i$ (not necessarily distinct) such that $x + y = z$. (This is known as Schur's Theorem.)

Brain Teaser 12.

(a) Find a set of eight points in the plane in general position so that no five of them are the vertices of a convex pentagon.

(b) Show that for any nine points in the plane in general position, some five of them are the vertices of a convex pentagon.

Tossing the Candle Out the Window

Brain Teaser 13.

(a) Show that $R(3, 5) = 14$.

(b) Show that $R(4, 4) = 18$.

Brain Teaser 14. Prove the generalisation of Ramsey's theorem we used to prove the Happy Ending Theorem: Suppose that at a party, every possible group of k people gets to talk with each other, and these groups were either loud or quiet. Then if we have enough people at the party, there will either be m people where every subset of k people talked loudly, or n people where every subset of k people talked quietly.

Brain Teaser 15. Show that $R(k, k) \geq 2^{\frac{k}{2}}$ for all positive integers $k \geq 3$. (This is basically the best general lower bound for Ramsey numbers that we have at the moment, and was also proven by Erdős.)