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Grade 7/8 Math Circles

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Group Theory

Rubik's Cube

We will begin to learn about Group Theory by looking at one of the famous toys in history, the Rubik's cube. Invented in 1974 by Erno Rubik of Budapest, Hungary, the Rubik's cube comes prepackaged in a solved position, where each face of the cube has the same colour. However, we can scramble the cube by rotating any one of it's six faces. The goal of this particular puzzle is to return the cube back to it's original/solved position.

The Rubik's cube is of significant mathematical interest because of it's symmetrical nature. Symmetry is present everywhere in mathematics, but nowhere as studied or observed than in **Group Theory**. Can you give or think of examples of symmetry?

In the Rubik's cube,

- There are a set of actions you perform on the cube (can rotate any of it's 6 sides)
- Each action can be reversed (can rotate the opposite way to undo rotations)
- Combining actions results in another action

Groups

Using what we learned about the Rubik's Cube, we will define a **group** as follows:

Group

For a nonempty set G and a list of defined actions on elements of G, G is a *group* if:

(Inverse Element) Every action is reversible by another action.

(Identity) There is an action that does nothing.

(Closure) Consecutive actions result in an action we previously defined.

Recall that an **integer** is a whole number that can be positive, negative or zero.

Example: Let's define the set of integers as {..., -3, -2, -1, 0, 1, 2, 3, ...} and the action defined here is the addition of any two integers. Let's prove that this is a group.

- *Inverse Element*: We must first show that every action is reversible by another action. Whenever we add an integer n, we can reverse the action by ______________________.
- *Identity*: Clearly if we add ______ to any integer, we are not changing anything so adding ______ is the identity action and ______ is the **identity element**.
- *Closure*: Adding any two integers together results in an __________ which is an element in our set so closure exists.

All required conditions are met so the set of integers with the action of addition is a **group**.

Non-Example: Given the set {2, 3, 4} and the action of multiplication. We know that for multiplication, if we multiply by 1 we change nothing so 1 is the **identity element** but 1 is not in this set of numbers so this is not a group as the Identity condition is not held.

Non-Example: Given the set of integers {..., -3, -2, -1, 0, 1, 2, 3, ...} and the action of multiplication, let's try to find the inverse element of multiplying by 2. That is, let's multiply $a \times 2$ by something to reverse the multiplication by 2 to get a. We know that we need to multiply by $\frac{1}{2}$ but $\frac{1}{2}$ is not in this set of integers so this is not a group as the Inverse Element Condition is not held.

Exercise Set 1

Recall that a **rational number** is a number that can be expressed as the fraction $\frac{a}{b}$ of two integers, a numerator a and a *non-zero* denominator b.

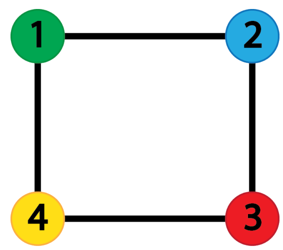
Exercise: Let $\mathbb{Q} \setminus 0$ be the set of all rational numbers **excluding 0** with the action of multiplication. Prove that this is a group.

Exercise: In the exercise above, why is the set of rational numbers, $\mathbb{Q}$ (which includes 0), with the action of multiplication not a group?

Exercise: Give two reasons why the set of integers {..., -3, -2, -1, 0, 1, 2, 3, ...} with the defined action of division is **not** a group.

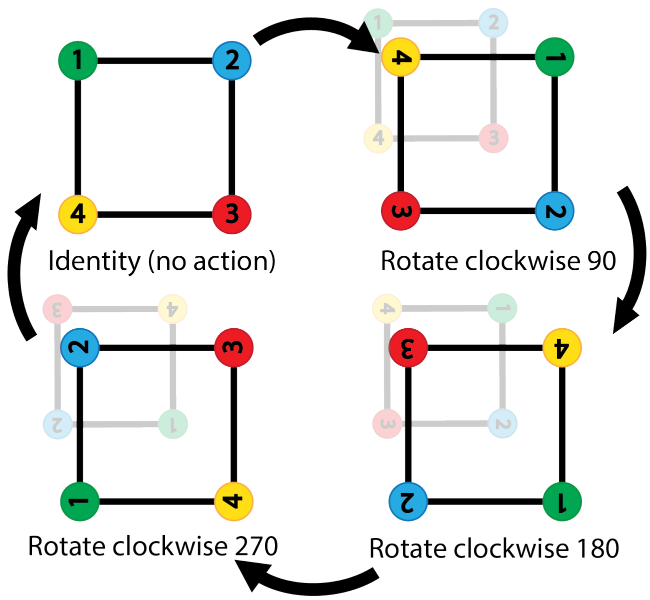
Rotations

Example 1: You are given the square below, with labels 1, 2, 3, 4 on the corners of the square and you are ONLY allowed to rotate it **clockwise** by 90°, 180°, and 270°.



We can see how this is a group:

- **4 actions**: do nothing (rotate 0°), rotate 90°, 180°, and 270° clockwise
- A rotation can be undone by another rotation. For example, if I rotate 90°, and then I rotate 270°, I'll return the square back to it's original position
- Two rotations combined is equivalent to another rotation.



To simplify the notation, we will the use the following to represent our actions:

$$\{\mathbf{I}, \mathbf{R_{90^\circ}}, \mathbf{R_{180^\circ}}, \mathbf{R_{270^\circ}}\}$$

We can use two actions to formulate a third action. For example, combining a rotation of 90 degrees and 180 degrees gives me a rotation of 270. We will write it as

$$R_{90^\circ} R_{180^\circ} = R_{270^\circ}$$

Exercise:

1. What is $R_{90^\circ} R_{90^\circ}$? Draw out how the square looks like after the two rotations.

2. What is $R_{180^\circ} R_{270^\circ}$? Draw out how the square looks like after the two rotations.

3. If I rotated 270° 53 times, what will my square look like at the end?

4. When we combine two rotations we always end up with another rotation. Does the order how you combine the rotation matter? For example, if I rotated 90° clockwise and then 180° is it the same as rotating 180° clockwise and then 90° clockwise? Justify your answer.

Example 2 - (A Non-Group):

If we are not careful with the actions we allow, it may not be a group! Using the same square, let's say we are only allowed two actions - **flipping vertically** and **flipping horizontally**. You may also assume we can do nothing as well. Let's denote them as f_v for flipping vertically and f_h as flipping horizontally. Is this a group?

If it's not a group, can we add an action to fix this?

Hint: Remember that two actions must combine to from our list of allowed actions. It may be helpful to draw out every combination of the two actions.

How might we fix this then?

We can add the action of: ____________

In total our four actions are now: ____________

Notice that reflecting horizontally and vertically is the equivalent as rotating the square 180°. So perhaps, we can just add a rotation of 180° as an action to our list, but we must check that when we combine a rotation of 180° with either a horizontal or vertical reflection, we get back one of our actions in our list.

Exercise: Check if the following actions on the square above form a group.

$$\{F_v, F_h, R_{180^\circ}, I\}$$

Organizing Group Actions: Cayley Tables

Drawing every possible combination of our permitted actions quickly becomes cumbersome. Instead, we can construct a square table to see all the possible combinations of actions performed on a square. This is called a **Cayley Table**.

Example.

Going back to our first example with the rotations. We can express all combinations succinctly the chart shown below.

Action	**I**	R_{90°	R_{180°	R_{270°
I				
R_{90°				
R_{180°				
R_{270°				

Exercise.

Construct the Cayley Table for Example 2 with our 3 actions in addition to doing nothing - Rotation Clockwise 180°, Vertical Reflection, and Horizontal Reflection.

More on Groups

There is more to groups than the definition. Below are some properties of groups.

> **Order of a Group**
>
> The **order** of a group is the number of allowed actions.

Example: In the first rotation example, the defined actions were: $\{I, R_{90^\circ}, R_{180^\circ}, R_{270^\circ}\}$. The **order** of this group is 4 as there are 4 actions allowed.

Exercise: What is the order of the final group in the second rotation example?

Exercise: * Recall the set of integers {...-3, -2, -1, 0, 1, 2, 3, ...} with the defined action of addition. What is the order of this group?

A **theorem** is a statement that can be demonstrated to be true by accepted mathematical operations and arguments. The process of showing a theorem to be correct is called a **proof**.

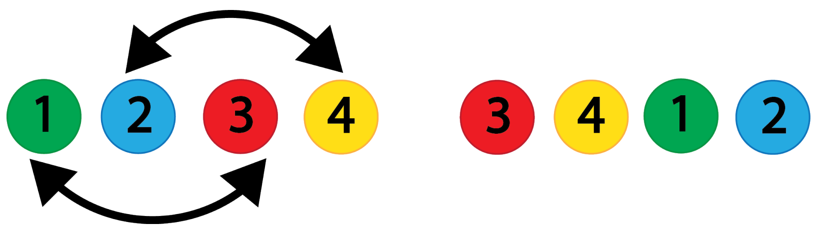
> **Theorem 1 - Uniqueness of Identity**
>
> For a defined group, there is only one **identity element** that doesn't do anything.

In other words, there is only one way to do nothing.

> **Theorem 2 - Uniqueness of Inverses**
>
> For a defined group, if b and c are both **inverses** of a then $b = c$.

Permutation Groups

Now let's observe another type of action we can do - rearranging the order of 4 balls. We call the different rearrangements, **permutations**. To rearrange or to permutate the order of our objects, we may swap the location of any two objects. For example, we have four balls, let's swap the 2^{nd} ball's location with the 4^{th} ball, and the 3^{rd} ball with the 1^{st} ball's location.



Exercise: Consider the set of 4 balls labelled 1, 2, 3, and 4 with the action of swapping the location of the balls. Is this a group? Prove why or why not.

Exercise: List all the possible different ways, you can arrange the 4 balls shown above.

Hint: It may be helpful to determine the total number of different arrangements first.

***Note**: You can only have one ball in each position.*

Example.

Suppose I have the 4 balls lined up from 1 to 4 in order. Instead of swapping, let's relocate each ball to a different position.

- I move the ball from the **first position** to the **fourth position**
- I move the ball from the **second position** to the **first position**
- I move the ball from the **third position** to the **second position**
- I move the ball from the **fourth position** to the **third position**

What does my final arrangement look like?

We write this mathematically as:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

Here we have an array of numbers, where the top row indicates the which position we are referring to initially, and the bottom number indicates which position we are sending the ball. For example, below the number 1 on the top row is 4. The ball that is located in position 1 is now placed in the fourth position. Similarly the ball in position 2 on it is now placed in the first position and so forth.

Exercise: What does my final arrangement look like for the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Permutation Groups - Multiple Rearrangements

Here is something more interesting, let's move every ball from a position **twice**. Suppose I have 4 balls as labelled above.

- I move the ball from the **first position** to the **second position**
- I move the ball from the **second position** to the **third position**
- I move the ball from the **third position** to the **fourth position**
- I move the ball from the **fourth position** to the **first position**

Now with the balls already moved once from their initial position. Let's move them again.

- I move the ball form the **first position** to the **third position**
- I move the ball from the **second position** to the **fourth position**
- I move the ball from the **third position** to the **second position**
- I move the ball from the **fourth position** to the **first position**

We write this mathematically as:

$$\overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}}^{\text{Second Rearrangment}} \overbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}}^{\text{First Rearrangement}}$$

When we combine permutations, we read from **right to left**. What does the final configuration look like?

Exercise:

For the following rearrangement actions, determine the equivalent action. and draw the final configurations of where the balls are.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$

Undoing the Rearrangement

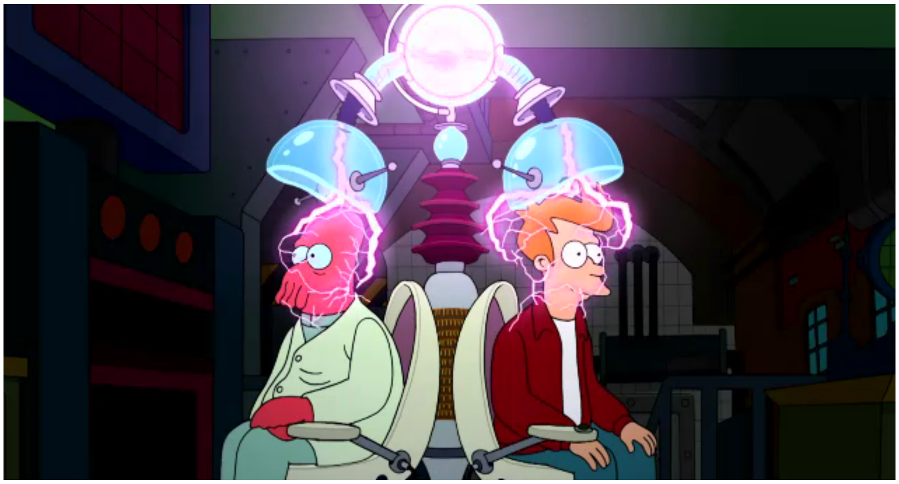
Suppose we are given the rearrangement rule in the array below, how can return all the balls back to it's original position?

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

1. Can you create another rearrangement rule that returns all the balls to their initial position?

2. Is it possible to keep applying the same rearrangement rule to return all the balls to their initial position?

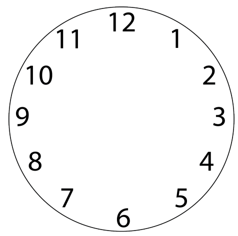
The Futurama Problem



An episode of Futurama, the prisoner of Brenda, received critical acclaim for popularizing math. In this episode, Professor Farnsworth and Amy build a machine that allows them to switch minds. However, the machine can only switch minds between two bodies only once, so they are unable to return to their bodies. In an attempt to return to their original bodies, they can invite other people to switch bodies with them. Is it possible for everybody to return to their original body? If so, how can this be done? How many people do they need to invite?

Problem Set

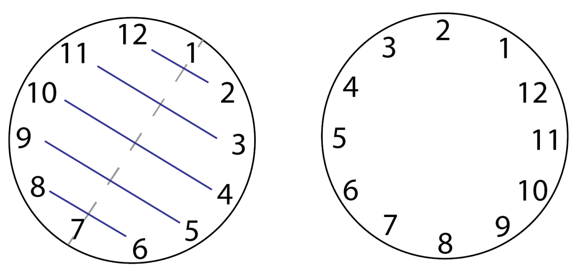
1. **Clock**. The clock is an interesting source of symmetry which naturally makes it of mathematical interest.



   (a) Suppose we can only rotate the clock by 1 hour. How many possible rotations are there?

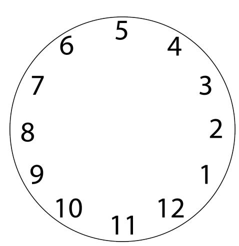
   (b) How many possible reflections are there? A reflection is done by drawing between two numbers on a clock diametrically opposite away from each other (equal distance away from other). For example 12 and 16 are diametrically opposite as well as 10 and 4. Then all the number reflect across that line.

   (c) If I combine to reflections together, what is their equivalent action?



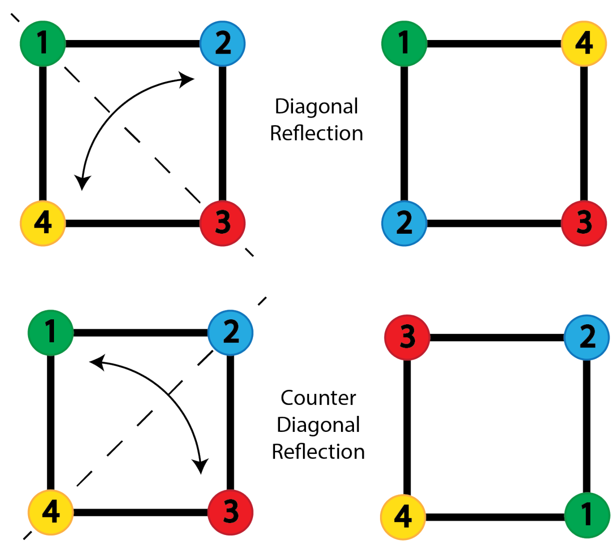
7 and 1 are diametrically opposite so the line from 7 to 1 is the line of reflection
blue line indicates which numbers are swapped

   (d) The clock below is scrambled. Can you using just rotations and reflections, return the clock back to it's normal face? How many actions do you require? Can you come up with multiple ways?



2. **The Light Switch**. Suppose we have two light switches one next to the other. You have the following actions - flipping the first switch, flipping the second switch, switching both switches, and as usual doing nothing. Draw all the possible configurations. Is this a group?

3. Using the square below (the same as the class example), but now we add a reflection diagonally



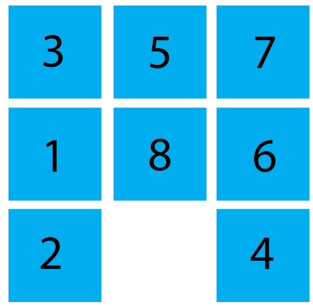
With the addition of these two actions (reflection diagonally) F_d and a reflection counter diagonally F_c, along side the actions we did in class i.e. rotate by 90° R_{90°, rotate by 180° R_{180°, rotate by 270° R_{270°, horizontal reflection F_h, and vertical reflection F_v. Draw out the Cayley Table. After seeing the Cayley Table, determine if this is a group.

4. Simplify multiple permutation actions as one equivalent permutation action and draw out the final configuration.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

5. Let $\mathbb{Q}$ be the set of all rational numbers with the action of addition. Prove that this is a group.

6. **Sliding Puzzle** In the sliding puzzle, there is vacant spot, you may move any block adjacent to the vacant space (either horizontally or vertically in the vacant spot). Is it possible to keep moving these places around given one space to arrange the 3×3 block into block that puts all the number in order?



7. **Three Cups Problem** We are given three cups. One cup is upside down, and the other two is right-side up. The objective is to turn all cups right-side up in no more than six moves. Each time, you must turn over exactly two cups per move. Is this possible?

8. **The $100 Prize** The principal of a school offers 100 students, who are numbered from 1 to 100, a chance to win $100 each. A room contains a cupboard with 100 drawers. The principal randomly puts one student's number in each closed drawer. The students enter the room, one after another. Each student may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every student finds his number in one of the drawers, all students win the $100 prize each. If just one student does not find his number, no student wins. Before the first student enters the room, the students may discuss strategy - but may not communicate once the first student enters to look in the drawers. What is the students' best strategy?