Faculty of Mathematics
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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

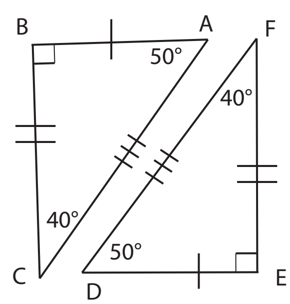
October 8/9/10 2019

The Math Circles with Triangles

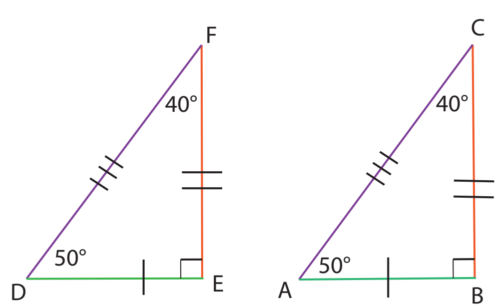
Triangles are simple yet complex polygons with three sides. This lesson we will go in depth into the shape and its many interesting properties.

Similar and Congruent Triangles:

Two triangles are **Congruent** if they have exactly the same three side-lengths and exactly the same three angles. Essentially the triangles must be identical to each other.
We say their **corresponding sides/angles** are equal.



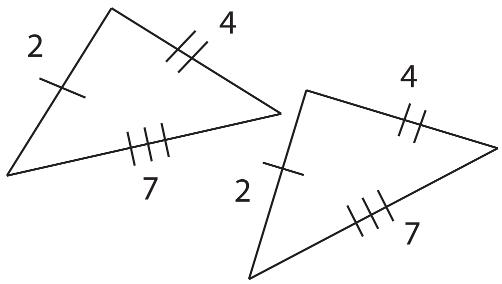
Corresponding Sides/Angles: "Matching" sides and angles that are in the same spot in two different shapes. Sometimes it can be harder to see corresponding sides/angles when the shapes are rotated.



With both triangles rotated to the same orientation we can see three pairs of corresponding sides (AB, DE), (BC, EF), (CA,FD) and three pairs of corresponding angles. ($\angle A$, $\angle D$) , ($\angle B$, $\angle E$) ($\angle C$, $\angle F$).

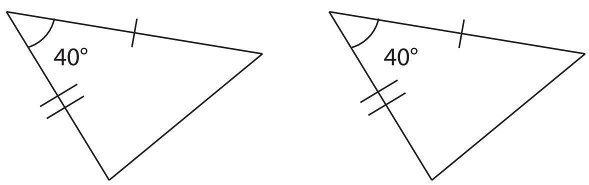
Checking for Congruency: There are several criteria we can check for to confirm if two triangles are congruent even if we are not told all the sides and angles. If just one of these criteria below is met then it can be said that a pair of triangles is congruent.

- Side-Side-Side



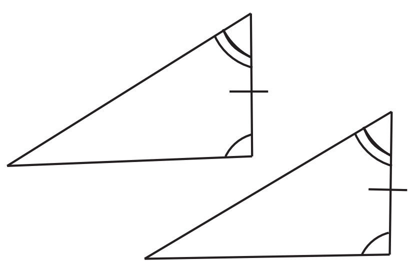
All three pairs of corresponding sides are the same length.

- Side-Angle-Side



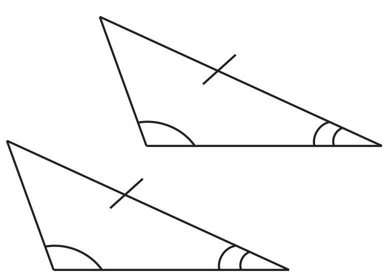
Two pairs of corresponding sides with the same length and the corresponding angles between them are equal.

- Angle-Side-Angle



Two pairs of equal corresponding angles and the corresponding sides between them are the same length.

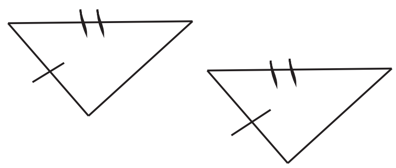
- Angle-Angle-Side

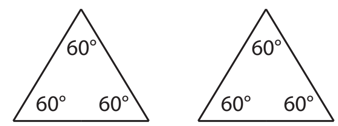


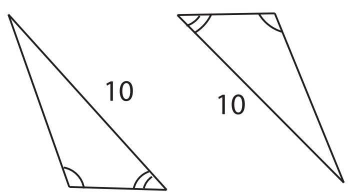
Two pairs of equal corresponding angles and a pair of corresponding sides (not between the angles) of the same length.

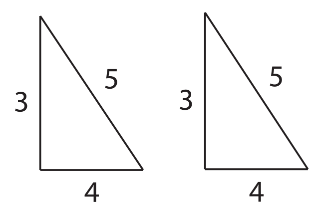
Examples:

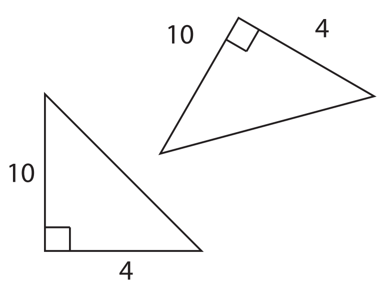
1.1 Which pairs of triangles can we conclude are congruent based on the criteria? Note that triangles are not drawn to scale.

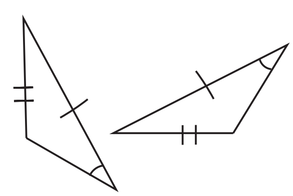






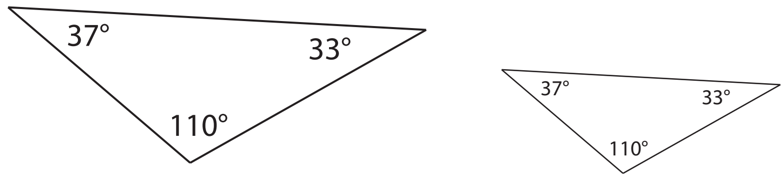






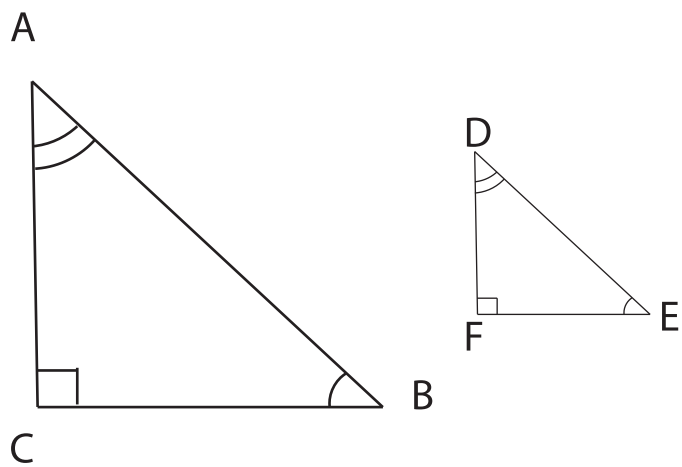
1.2 Is Side-Side-Angle enough to show two triangles are congruent? (Side-Side-Angle: Knowing two pairs of corresponding sides and a pair of corresponding angles not between them.)

Two triangles are **Similar Triangles** if their corresponding angles are equal. Basically, similar triangles are a similar shape but not necessarily the same size.



The pair of triangles above are **similar** as they share the same angles.

<u>Exercise:</u> Measure the sides of the following pair of similar triangles.



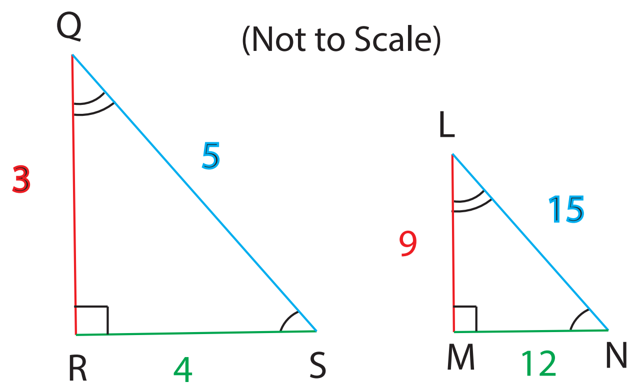
Side	***Length***
AB	
BC	
CA	

Side	***Length***
DE	
EF	
FD	

What do you notice about the relationship between the corresponding sides of $\triangle ABC$ and $\triangle DEF$?

We can say that $\triangle ABC$ has sides that are scaled larger than $\triangle DEF$. We call this **proportionate sides**. Thus we can say that our similar triangles have proportionate corresponding sides.

In fact all similar triangles have proportionate corresponding sides.



In the figure above we can see $\triangle LMN$ has proportionate sides (labelled by colour) that are three times as large as $\triangle QRS$. For example $MN = 3 \times RS$. **Thus, we can say that there is a scale factor or scale ratio of 3 or 1:3** between $\triangle QRS$ and $\triangle LMN$.

$$\mathbf{3} \times QR = LM$$
$$\mathbf{3} \times RS = MN$$
$$\mathbf{3} \times NL = SQ$$

All corresponding pairs are proportionate by a factor of 3.

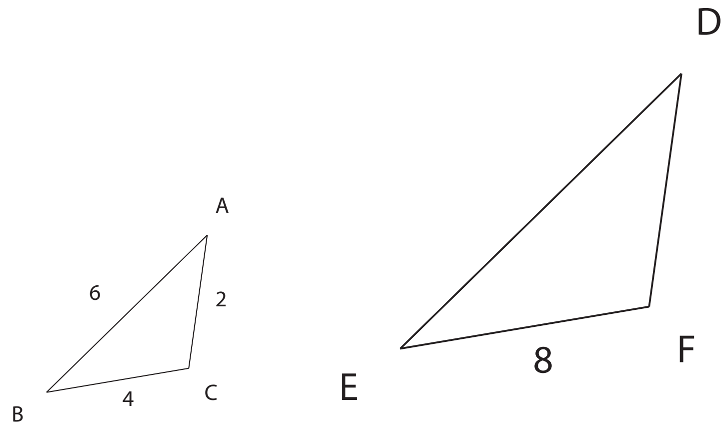
We can find a scale factor or ratio between any two pairs of similar triangles.For any pair of similar triangles we have an equation of the form:

$$\frac{LM}{QR} = \frac{MN}{RS} = \frac{NL}{SQ} = Scale\ Factor$$

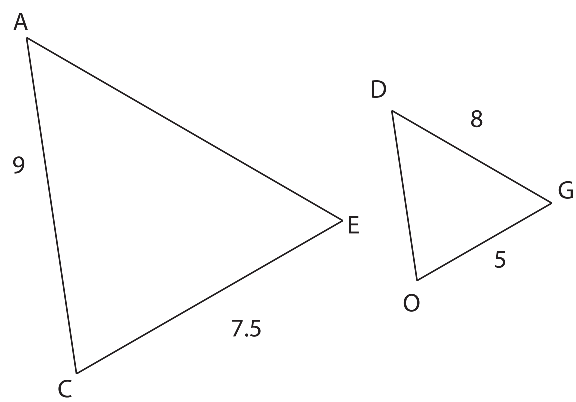
Where each fraction is a pair of corresponding proportionate sides.

Examples:

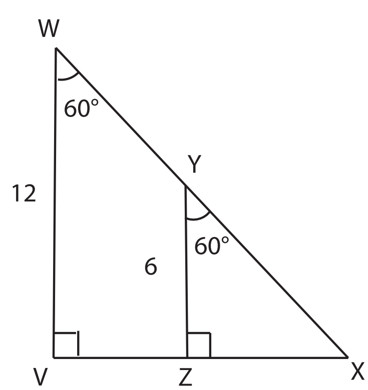
1.3 With the knowledge that $\triangle$ABC and $\triangle$DEF are similar triangles what are the missing side-lengths FD and DE?



1.4 $\triangle ACE$ and $\triangle DOG$ are similar. Find the missing side-lengths.



1.5 Can we say that $\triangle WVX$ and $\triangle XYZ$ similar? Explain. If VX= 16 and XW=20, find the missing side-lengths of $\triangle XYZ$.



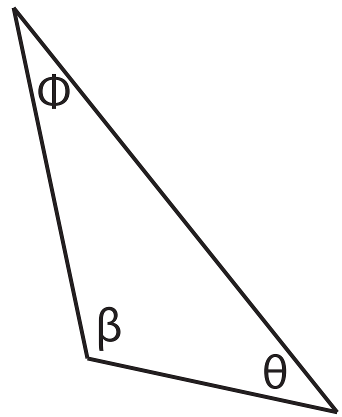
1.6 Are all similar triangles congruent? Are all congruent triangles similar?

Sum of Angles:

Another important property of triangles is that the sum of all their interior angles is 180°. This is useful in solving for missing angles.

Looking at Triangle 1 we would know that:

$$\phi + \beta + \theta = 180^\circ$$



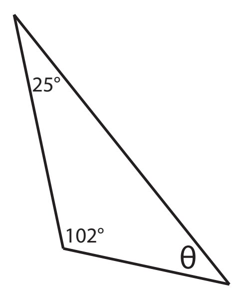
Triangle 1

In the case where we know angles β and ϕ we can use this information to solve for the missing angle θ.

As an example let's say $\beta = 102^\circ$ and $\phi = 25^\circ$ as shown in Triangle 2. Using the above equation we can say that $25^\circ + 102^\circ + \theta = 180^\circ$.Then using some algebra we can find our missing angle:

We can isolate for θ by subtracting 102° and 25° on both sides of the equation:

$$\begin{aligned} 25^\circ + 102^\circ + \theta &= 180^\circ \\ 25^\circ + 102^\circ + \theta - 25^\circ - 102^\circ &= 180^\circ - 25^\circ - 102^\circ \\ \theta &= 180^\circ - 25^\circ - 102^\circ \\ \theta &= 180^\circ - 127^\circ \\ \theta &= 53^\circ \end{aligned}$$



Triangle 2

Angles:

Let's stay on this topic of angles and discuss **Complementary** and **Supplementary** angles.

Complementary Angles are two angles that add to 90 degrees.

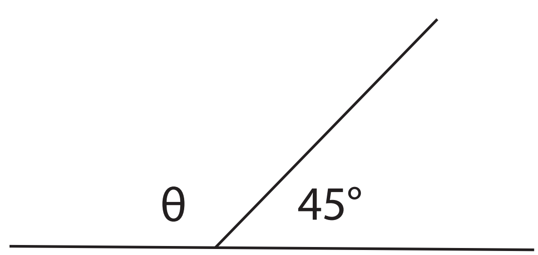
In the figure above we see how angles a and b form a right angle. Therefore we can say that angles a and b are complementary and thus: $a + b = 90°$

Supplementary Angles are two angles that add up to 180 degrees.

In the figure above we can see how angles a and b connect to form a semicircle and an angle of 180°. Therefore we can say that a and b are supplementary and thus $a + b = 180°$.

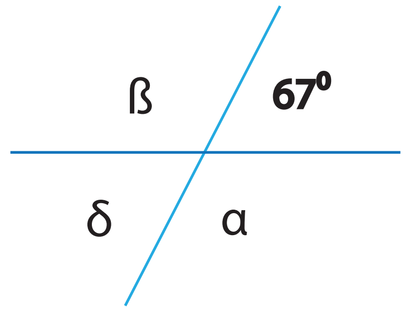
Practice:

How can we solve for θ in the figure?

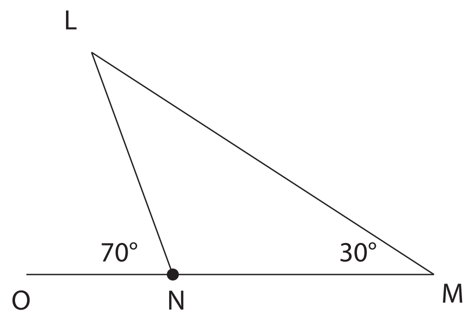


Examples:

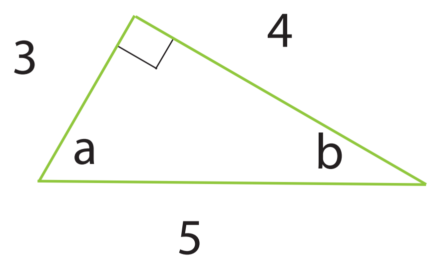
2.1 Find the missing angles β, δ and α in the figure.



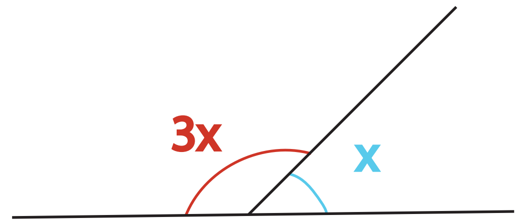
2.2 Find missing angles $\angle$LNM and $\angle$NLM.



2.3 Are angles a and b complementary, supplementary or neither? Explain.

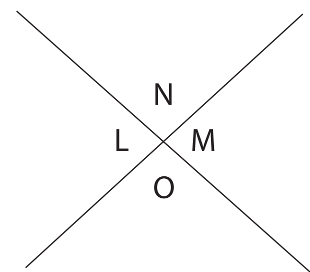


2.4 Solve for x:



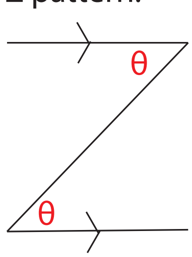
Think...

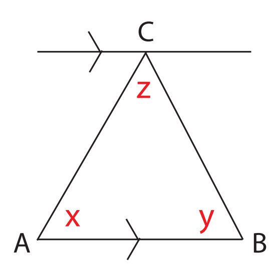
2.5 In the diagram below $\angle L = \angle M$. Also, $\angle N = \angle O$. (Opposite angles are equal.)Explain why this is true for all X patterns with your knowledge of supplementary angles.



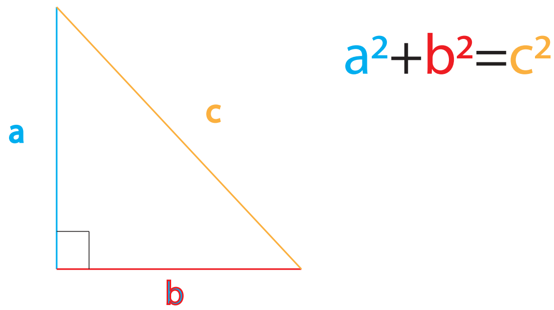
2.6 Using the Z pattern shown below, show that the sum of angles in $\triangle ABC$ is 180 degrees. (The Z pattern says that the alternate angles shown as θ in this case are equal.)

Z pattern:



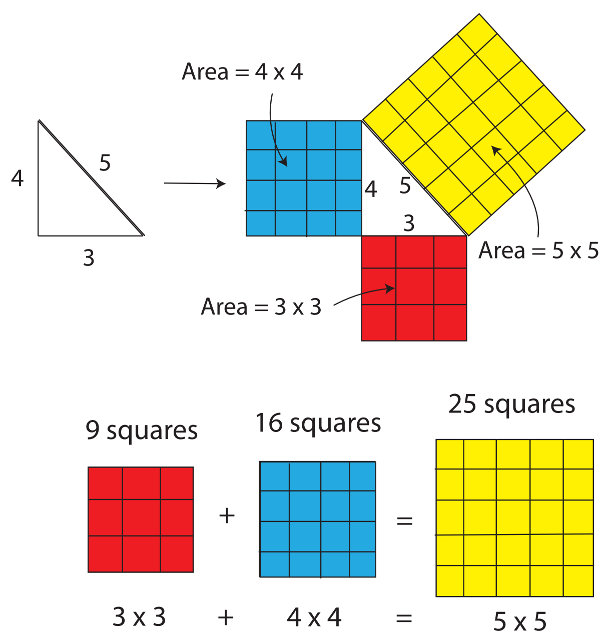


__**Pythagorean Theorem:**__ An additional tool in geometry and our final one for this lesson is the famous Pythagorean theorem. It relates the three side lengths of a **right angled triangle** with the formula:



The theorem says that squaring the smaller sides of the triangle and adding them together is equal to the square of the longest side-length of the triangle. **Note:** $a^2 = a \times a$
‘

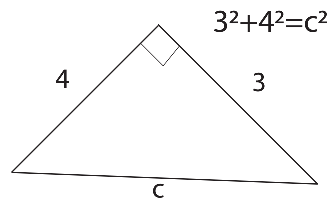
The geometrical interpretation can be seen below for a triangle of side-lengths a,b and c: If we expand the triangle to attach squares to each side we see something interesting...



The idea is that if we combined the area of the smaller red and blue squares they would be equal to the area of the biggest yellow square. Thus giving us the famous equation $a^2 + b^2 = c^2$

Using Algebra:

In order to find c in the triangle to the right we would use Pythagorean theorem as seen above. Adding $3^2 + 4^2$ gives us the square of the longest side (c^2.) We want to solve for c rather than c^2 so we are required to do some algebra:

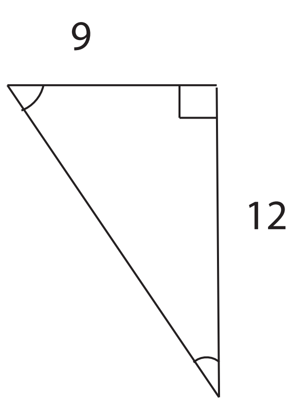


$$\mathbf{3^2 + 4^2 = c^2 \rightarrow 3 \times 3 + 4 \times 4 = c^2} \tag{1}$$
$$\mathbf{9 + 16 = c^2} \tag{2}$$
$$\mathbf{25 = c^2} \tag{3}$$
$$\textit{Want to solve for c not } c^2$$
$$\mathbf{\sqrt{25} = \sqrt{c^2}} \tag{4}$$
$$\textit{Square root } (\sqrt{}) \textit{ is the opposite operation to } ^2 \textit{ (squaring) so they cancel}$$
$$\textit{Thus } \sqrt{c^2} = c \ (\textit{for } c > 0)$$
$$\textit{Continuing from } (4): \mathbf{\sqrt{25} = c} \tag{5}$$
$$\mathbf{5 = c} \tag{6}$$

We find that the longest side of the triangle is 5 units.

Note this was the side-length of the yellow square we pictured on the previous page! Our result from algebra lines up with our illustration.

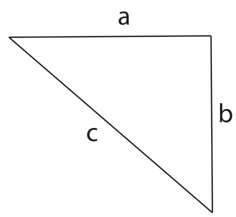
Practice: Find the missing side of the triangle below:



Examples:

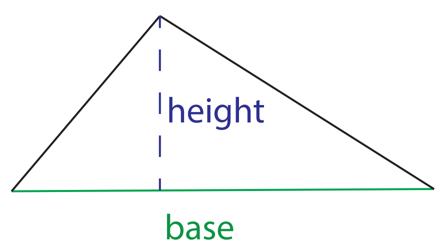
3.1 Determine the length of the diagonal of a square with area 4cm^2

3.2 Would it be valid to perform $c^2 + b^2$ to find a^2 on the triangle below?



Review: Area of a Triangle

The Area of the triangle can be calculated by the formula:

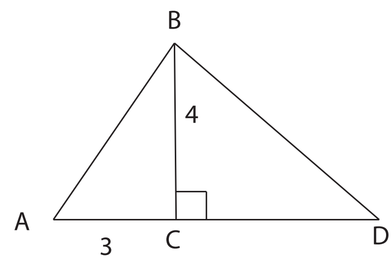


$$A = \frac{(base \times height)}{2}$$

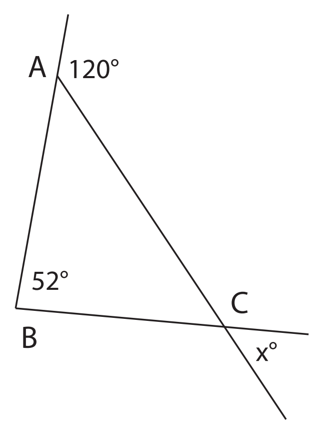
Problem Set

Now that we are all masters of triangles let's try some challenge questions that require the use of multiple tricks we covered today:

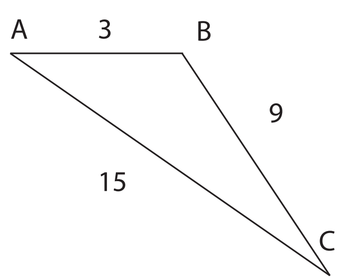
P.1 AC is half the length of CD. Find the area and perimeter of $\triangle$ABD.



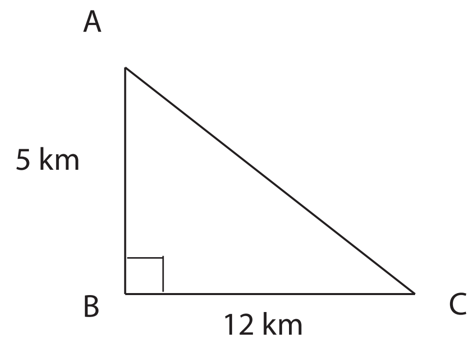
P.2 Find the measure of $\angle x°$



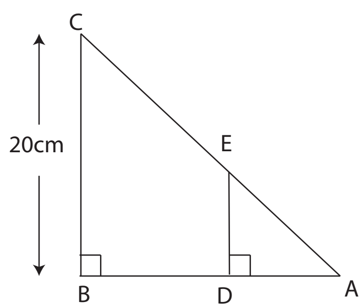
P.3 $\triangle ABC$ is similar to $\triangle DEF$. The sides of $\triangle DEF$ are proportionately larger by a scale factor of 4/3. What are the sidelengths of $\triangle DEF$? Sketch $\triangle DEF$.



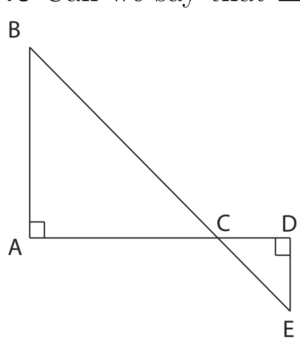
P.4 Cindy and Steph are each biking. Cindy bikes in a straight line from A to B, then bikes in a straight line from B to C. Steph bikes in a straight line from A to C. Who bikes more distance and by how much?



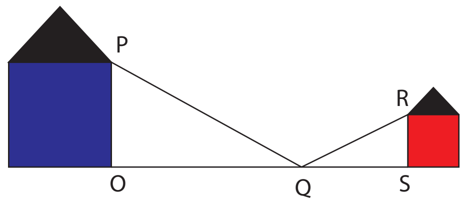
P.5 if $AE = 5cm$ and the area of $\triangle ABC$ is $150cm^2$ find the area of $\triangle$ADE.



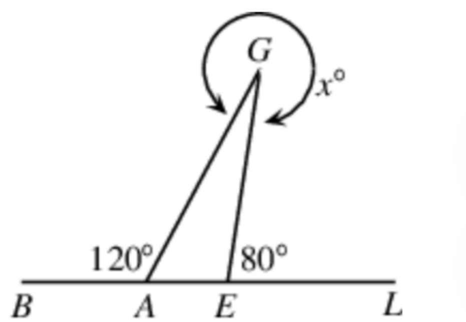
P.6 Can we say that $\triangle ABC$ and $\triangle CDE$ are similar from the figure?



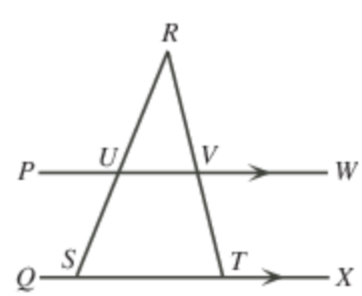
P.7 A blue house is 9m tall without the roof and a red house is 3m tall without the roof. The blue house is 40m from Point Q. Using the fact that △POQ and △QRS are similar find the length of a telephone cable run between the houses from R to Q and Q to P.



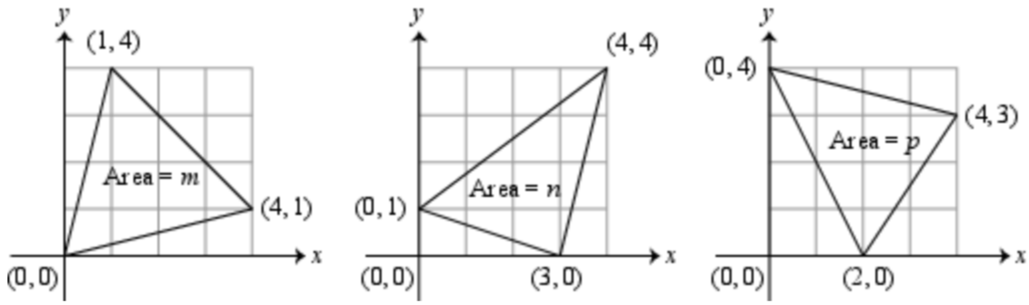
P.8 Four Points B , A, E, L are on a straight line, as shown. G is a point of the line so that $\angle BAG = 120°$ and $\angle GEL = 80°$. find reflex angle x. Hint: If there are 360° in a circle how are $\angle AGE$ and $\angle x$ related?



P.9 In the diagram, PW is parallel to QX, S and T lie on QX, and U and V are the points of intersection of PW with SR and TR, respectively. If $\angle PUR = 120°$ and $\angle VTX = 112°$, what is the measure of ∠ URV?



P.10 Each diagram shows a triangle labelled with its area. Calculate the areas m, n and p.



P.11 In the right-angled $\triangle PQR$, $PQ = QR$. The segments QS, TU and VW are perpendicular to PR, and the segments ST and UV are perpendicular to QR, as shown. What fraction of $\triangle PQR$ is shaded?