UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

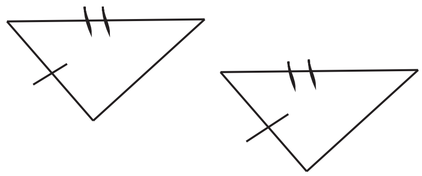
Grade 7/8 Math Circles

October 8/9/10 2019

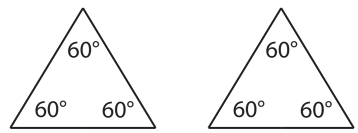
The Math Circles with Triangles Solutions

__Exercises:__

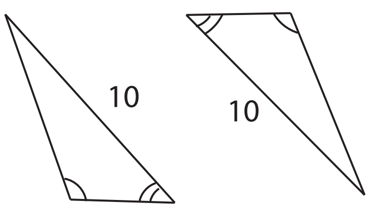
1.1 Which pairs of triangles can we conclude are congruent based on the criteria? Note that triangles are not drawn to scale.



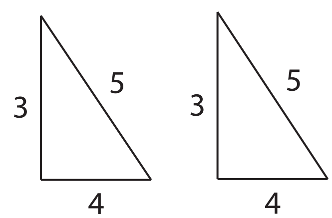
No, 2 matching sides does not meet criteria.



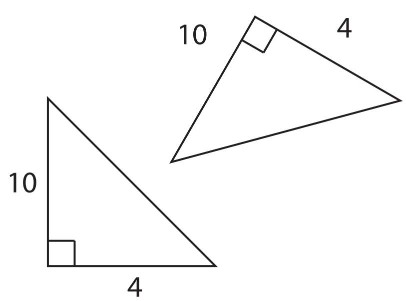
No, 3 matching angles does not meet cirteria. Equilateral triangles can be different sizes for example.



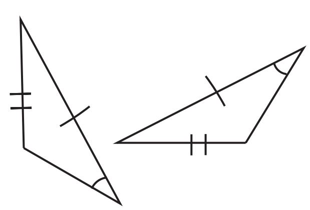
Yes, Angle-Angle-Side.



Yes, Side-Side-Side.



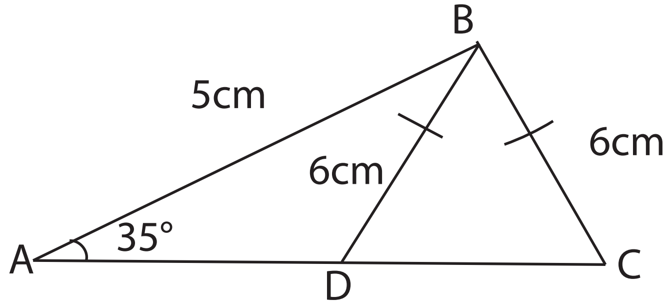
Yes, Side-Angle-Side.



No, Side-Side-Angle not valid criteria.

1.2 Is Side-Side-Angle enough to show two triangles are congruent? (Side-Side-Angle indicates we know two pairs of corresponding sides are equal as well as a pair of corresponding angles that is not between the sides.)

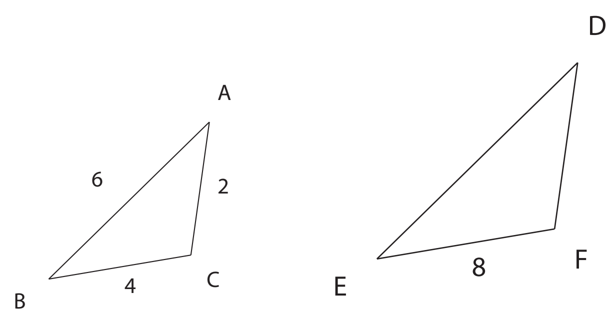
There are cases where multiple triangles can be formed if the remaining side and two angles are free to change in this criteria.



In the figure above we see an example of how we can form two non congruent triangles $\triangle ABC$ and $\triangle ABD$ that still meet this criteria.

Examples:

1.3 With the knowledge that $\triangle$ABC and $\triangle$DEF are similar triangles what are the missing side-lengths FD and DE?



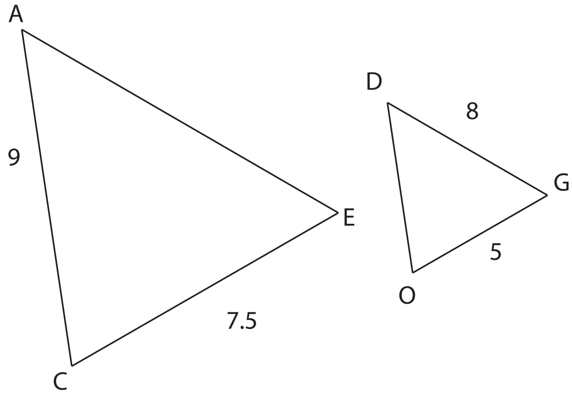
We know all the sides of $\triangle ABC$ and want to find the missing sides on $\triangle DEF$. Corresponding sides have a scale factor between them that we can find with the pair of corresponding sides BC and EF.

$$Scale\ Factor = \frac{EF}{BC} = \frac{8}{4} = 2$$

Therefore $\triangle DEF$ has corresponding sides twice as large as $\triangle ABC$.

$$FD = 2 \times AC = 2 \times 2 = 4$$
$$DE = 2 \times AB = 2 \times 6 = 12$$

1.4 $\triangle ACE$ and $\triangle DOG$ are similar. Find the missing side-lengths.



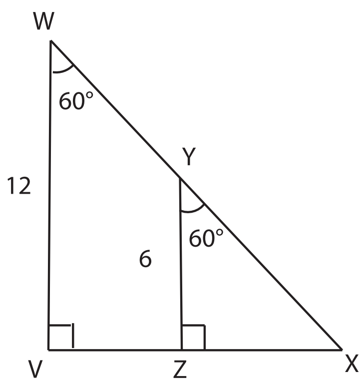
Using the pair of corresponding sides OG and CE we can find the scale factor between the two triangles. $Scale\ Factor = \frac{CE}{OG} = \frac{7.5}{5} = 1.5$

The sides on $\triangle ACE$ are scaled 1.5 or 3/2 times larger than $\triangle DOG$

We need to find missing sides AE and DO. Using the scale factor we know AE is 1.5 times larger than corresponding side DG and side DO is 1.5 times larger than corresponding side AE.

$$AE = 1.5 \times DG = 1.5 \times 8 = 12$$
$$DO = \frac{AC}{1.5} = \frac{9}{1.5} = 6$$

1.5 Can we say that $\triangle WVX$ and $\triangle XYZ$ similar? Explain. If VX= 16 and XW=20. Find the missing side-lengths.



The two triangles are similar as the two triangles share the same angles. (60,90, and both triangles share $\angle X$)

Same as before we must find the scale factor between $\triangle VWX$ and $\triangle XYZ$. Using the pair of corresponding sides WV and YZ:

$$Scale\ Factor = \frac{WV}{YZ} = \frac{12}{6} = 2$$

The sides on $\triangle WVX$ are scaled 2 times larger than $\triangle XYZ$

Using the scale factor we can find the missing sidelengths ZX and YX.

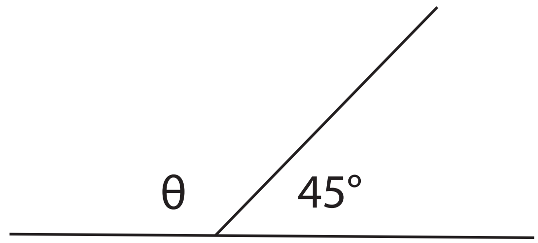
$$YX = \frac{XW}{2} = \frac{20}{2} = 10$$
$$ZX = \frac{VX}{2} = \frac{16}{2} = 8$$

1.6 Are all similar triangles congruent? Are all congruent triangles similar?

All congruent triangles are similar (have three equal corresponding equals) but not all similar triangles are congruent (different side lengths.)

Angles Practice:

How can we solve for θ in the figure?



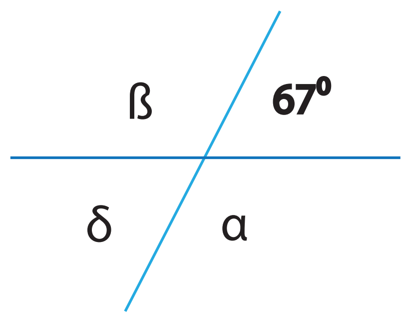
θ and $45°$ are supplementary angles. Thus we can say that: $\theta + 45° = 180°$

If θ and $45°$ add to $180°$ then subtracting $45°$ from $180°$ will leave us with just our angle θ

$\theta + 45° - 45° = 180° - 45°$
$\theta = 180° - 45° = 135°$

Angles Examples:

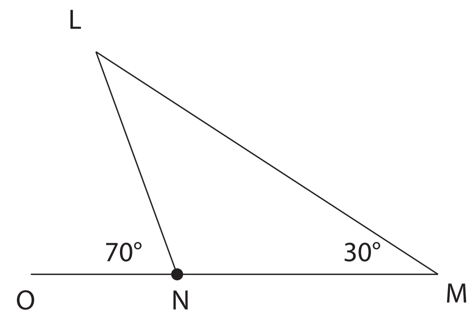
2.1 Find the missing angles β, δ and α in the figure.



Can identify the supplementary angles in the figure.
β and $67°$ are supplemental angles. Thus
$\beta + 67° = 180°$
$\beta = 180° - 67°$
$\beta = 113°$

$67°$ and α are also supplementary. Could show that $\alpha = 113°$ the same way as above.
δ is supplementary with β and α
$\delta + 113° = 180°$
$\delta = 180° - 113°$
$\delta = 67°$

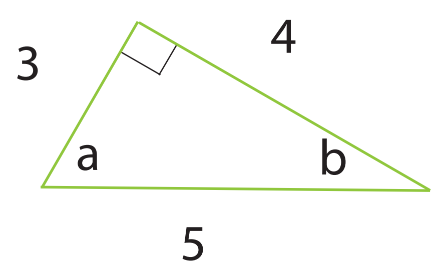
2.2 Find missing angles $\angle LNM$ and $\angle NLM$.



$\angle LNM$ and $70°$ are supplementary angles.
$\angle LNM + 70° = 180°$
$\angle LNM = 180° - 70° = 110°$

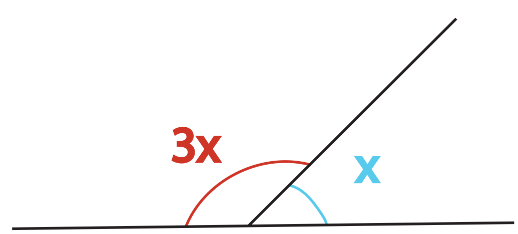
We know that all the angles in $\triangle LNM$ add up to $180°$.
$\angle NLM + \angle LMN + \angle LNM = 180°$
$\angle NLM + 30° + 110° = 180°$
$\angle NLM = 180° - 110° - 30°$
$\angle NLM = 40°$

2.3 Are angles a and b complementary, supplementary or neither? Explain.



We know from triangle properties that the sum of angles in a triangle is $180°$
$\angle a + \angle b + 90° = 180°$ $\angle a + \angle b = 180° - 90°$
$\angle a + \angle b = 90°$
Since the third angle in the triangle is $90°$ that leaves $180° - 90° = 90°$ for the remaining angles $\angle a$ and $\angle b$ Therefore $\angle a + \angle b = 90°$ making a and b complementary.

2.4 Solve for x:

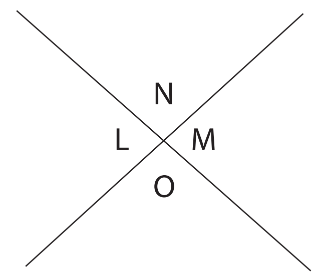


Angles are supplementary so we have equation:
$3x + x = 180°$
$4x = 180°$
This equation reads 4 times some angle 'x' is equal to $180°$. To find the measure of the angle we can divide $180°$ by 4.
From the above equation:
$\frac{4x}{4} = \frac{180°}{4}$
$x = 45°$

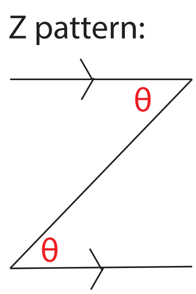
Think...

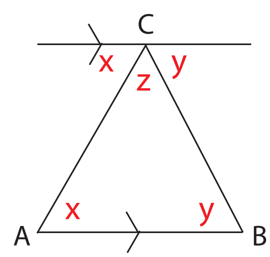
2.5 In the diagram below $\angle L = \angle M$. Also, $\angle N = \angle O$. (Opposite angles are equal.)Explain why this is true for all X patterns with your knowledge of supplementary angles.

Looking at $\angle O$ we could say that it is supplemental with angles L and angle M. Therefore L and M must be the same.
Mathematically this looks like:
$\angle O + \angle M = 180$
$\angle O + \angle L = 180$
Thus we could set both of these equations equal:
$\angle O + \angle M = \angle L + \angle O = 180$
The only way that these equations are satisfied is if $\angle L = \angle M$
Similarly we can show that $\angle N = \angle O$



2.6 Using the Z pattern shown below, show that the sum of angles in $\triangle ABC$ is 180 degrees. (The Z pattern says that the alternate angles shown as θ in this case are equal.)

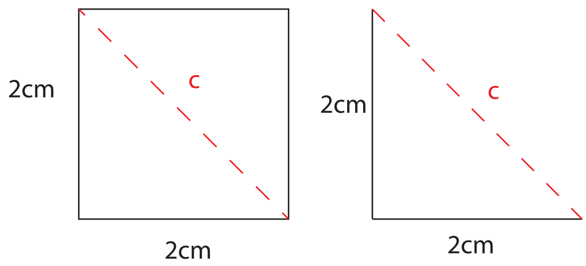




Using the Z pattern we find that we have angles x and y on either side of $\angle z$ in $\triangle ABC as shown above$
Since we have angles x, y and z forming a straight line we can say that they must add to 180°.
$\angle x + \angle y + \angle z = 180°$.
Thus we have shown all of the angles in $\triangle ABC$ must add to 180°.

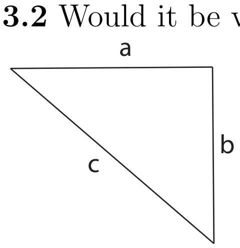
Pythagorean Theorem Examples:

3.1 Determine the length of the diagonal of a square with area 4cm^2



A square with area of $4cm^2$ must have side lengths of $2cm$ ($Area = side \times side$)
We are interested in finding the length of the red dashed diagonal of the square. Call the diagonal c.
The dashed line forms a triangle with two sides of the square.
We can use pythagorean thereom to find the length of the dashed line:
$2^2 + 2^2 = c^2$
$4 + 4 = c^2$
$8 = c^2$
$\sqrt{8} = \sqrt{c^2}$
$2.828.... = c$

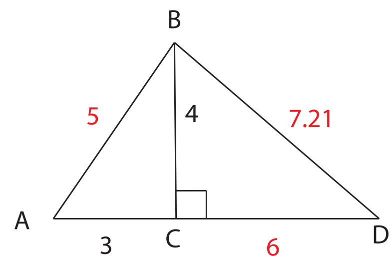
3.2 Would it be valid to perform $c^2 + b^2$ to find a^2? ($c^2 + b^2 = a^2$) on the triangle below.



It would not make sense as pythagorean thereom sums the squares of the two smaller sides of the triangle to find the longest side. In this equation we are adding the square of the longest side c to the square of one of the shorter sides b ($c^2 + b^2$) to get the length of a smaller side (a^2.) If we tried to find a this way we would find a length for a longer than c which is the longest side of the triangle. Thus this equation is not valid mathematically.

Problem Set:

P.1 AC is half the length of CD. Find the area and perimeter of △ABD.



$CD = 2 \times AC = 2 \times 3 = 6$

Can find area of $\triangle ABD$ as we know the base and height of $\triangle ABD$

base = AD = 3+6=9

$Area = (base \times height)/2 = (9 \times 4)/2 = 18$

To find perimeter of $\triangle ABD$ we need sides AB and BD.We can find them with pythagorean theorem in $\triangle ABC$ and $\triangle BCD$

In $\triangle ABC$:

$AB^2 = AC^2 + BC^2 = 3^2 + 4^2$

$AB^2 = 25$

$AB = 5$

In $\triangle BCD$:

$BD^2 = BC^2 + CD^2 = 4^2 + 6^2$

$BD^2 = 52$

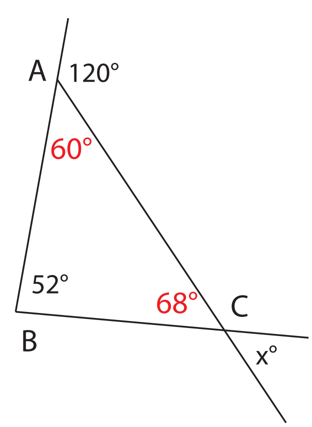
$BD \approx 7.21...$

$Perimeter = AB + BD + AD$

$Perimeter = 5 + 7.21 + 9$

$Perimeter = 21.21$

P.2 Find the measure of $\angle x°$



$\angle A$ and 120° are supplementary $\angle A + 120° = 180°$

$\angle A = 180° - 120° = 60°$

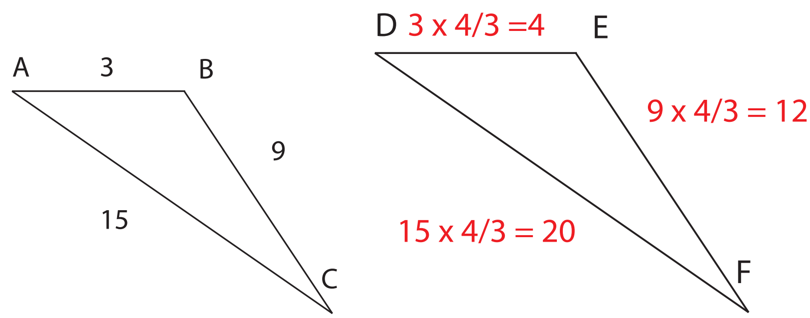
Can find $\angle C$ using the sum of angles in a triangle.

$60° + 52° + \angle C = 180°$

$\angle C = 180° - 60° - 52° = 68°$

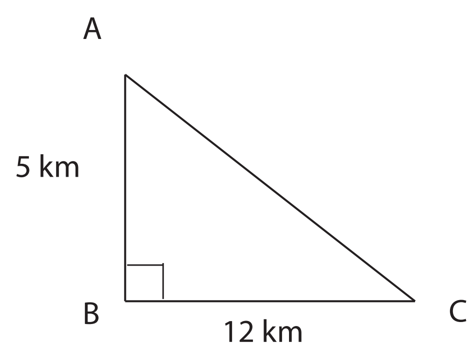
Using "X Pattern" from example 2.5 we can then say that $x = 68°$ (using supplementary angles.)

P.3 $\triangle ABC$ is similar to $\triangle DEF$. The sides of $\triangle DEF$ are proportionately larger by a scale factor of 4/3. What are the sidelengths of $\triangle DEF$? Sketch $\triangle DEF$.



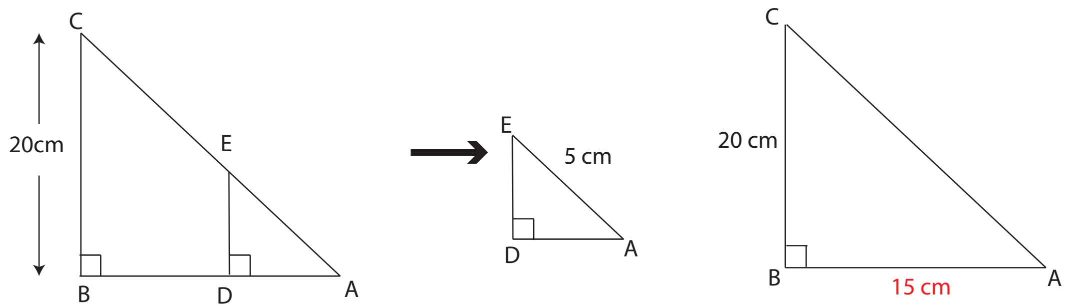
Multiply corresponding sides by the scale factor to obtain sidelengths for $\triangle DEF$. △ DEF is the same "shape" as △ ABC.

P.4 Cindy and Steph are each biking. Cindy bikes in a straight line from A to B, then bikes in a straight line from B to C. Steph bikes in a straight line from A to C. Who bikes more distance and by how much?

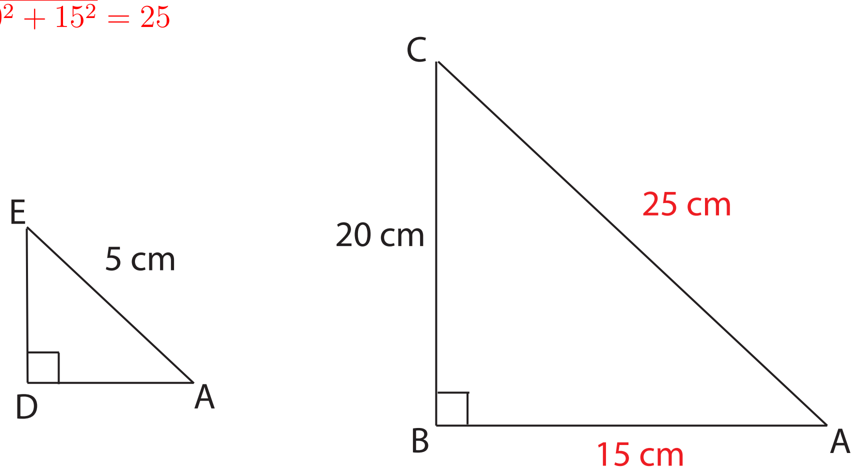


Cindy Distance: $5km + 12km = 17km$
Steph Distance: A to C. We can find the distance of AC using Pythagorean Theorem.
$AC = \sqrt{5km^2 + 12km^2}$
$AC = 13km$
Cindy bikes 17 km and Steph bikes 13km. Cindy bikes 4km farther than Steph.

P.5 if $AE = 5cm$ and the area of $\triangle ABC$ is $150cm^2$ find the area of $\triangle ADE$.
We can seperate the two triangles in the figure.



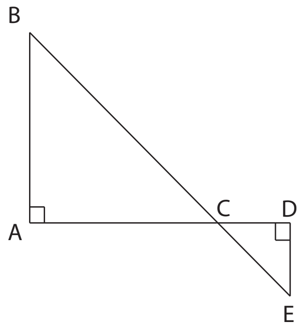
Area of $\triangle ABC = \dfrac{base \times height}{2} = \dfrac{AB \times 20}{10} = 150$
Rearranging the above equation we can solve for side AB:
$AB = \dfrac{150 \times 10}{20} = 15$ We can then find the side AC using pythagorean theorem:
$AC = \sqrt{20^2 + 15^2} = 25$



We can use the pair of corresponding sides AC and AE to find the scale factor between similar triangles $\triangle ABC$ and $\triangle ADE$.
$Scale\ Factor = \dfrac{AC}{AE} = \dfrac{25cm}{5cm} = 5$

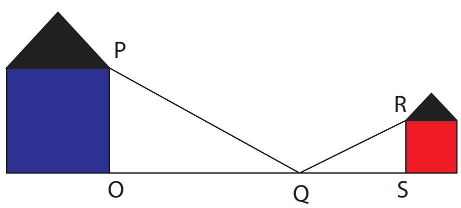
With the scale factor sides AD and DE can then be found:
$AD = \dfrac{AB}{5} = \dfrac{15cm}{5} = 3cm$
$DE = \dfrac{BC}{5} = \dfrac{20cm}{5} = 4cm$
Area of $\triangle ADE = \dfrac{base \times height}{2} = \dfrac{AD \times DE}{2} = \dfrac{3 \times 4}{2} = 6cm^2$

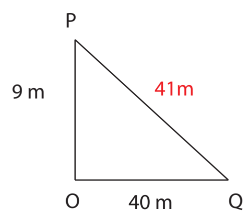
P.6 Can we say that $\triangle ABC$ and $\triangle CDE$ are similar from the figure?

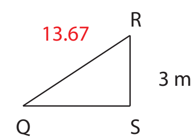


From the figure we can see that $\angle BAC$ and $\angle CDE$ are equal right angles.
From the X pattern discussed in **2.5** we can say that the opposite angles $\angle ACB$ and $\angle DCE$ equal.
From the sum of angles, if a triangle has two known angles the third angle can be only one value. Thus since we know $\angle BAC = \angle CDE$ and $\angle ACB = \angle DCE$, $\angle DEC$ and $\angle ABC$ must be equal.
$\triangle ABC$ and $\triangle CDE$ then have 3 pairs of equal angles meaning they are similar.

P.7 A blue house is 9m tall a red house is 3m tall. The blue house is 40m from Point Q. Using the fact that △POQ and △QRS are similar find the length of a telephone cable run between the houses from R to Q and Q to P.





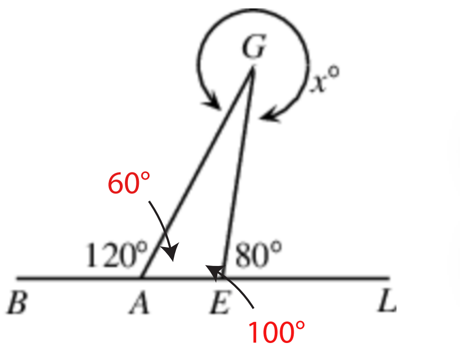


We can use pythagorean theorem to find PQ : $PQ = \sqrt{9m^2 + 40m^2} = 41m$

We can find the scale factor between $\triangle OPQ$ and $\triangle QRS$ using corresponding sides RS and OP :
$Scale\ Factor = \frac{OP}{RS} = \frac{9}{3} = 3$
QR can then be found from the scale factor.
$QR = \frac{41m}{3} = 13.67$

$Length\ of\ Telephone\ Pole = PQ + QR = 41m + 13.67m = 54.67m$

P.8 Four Points B , A, E, L are on a straight line, as shown. G is a point of the line so that $\angle BAG = 120°$ and $\angle GEL = 80°$. find reflex angle x. Hint: If there are 360° in a circle how are $\angle AGE$ and $\angle x$ related?

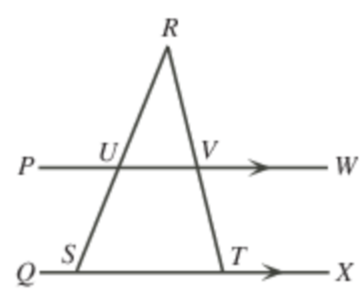


Since the sum of the angles at any point on a line is 180°, then $\angle GAE = 180° - 120° = 60°$ and $\angle GEA = 180° - 80° = 100°$.

Since the sum of the angles in a triangle is 180° :
$\angle AGE = 180° - \angle GAE - \angle GEA = 180° - 60° - 100° = 20°$.

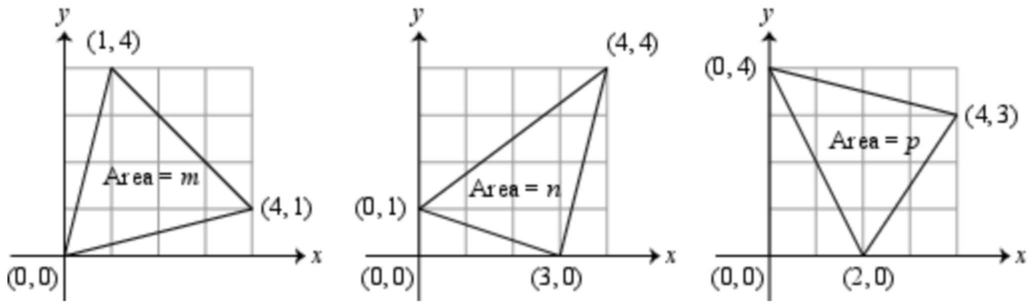
Since $\angle AGE$ and $\angle x$ form a full circle they must add to 360°. Thus $\angle x$ is the left over angle after taking away $\angle AGE$ (20°) from the circle.
$\angle x = 360° - 20° = 340°$.

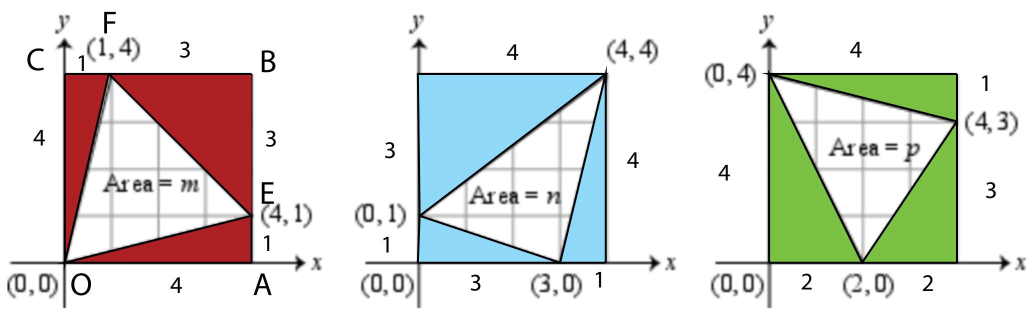
P.9 In the diagram, PW is parallel to QX, S and T lie on QX, and U and V are the points of intersection of PW with SR and TR, respectively. If $\angle PUR = 120°$ and $\angle VTX = 112°$, what is the measure of $\angle$ URV?



$\angle PUR$ is opposite of $\angle SUV$ thus we can say that $\angle PUR = \angle SUV = 120°$
Since SUR is a straight line, $\angle SUV$ and $\angle RUV$ are supplementary, then
$\angle RUV = 180° - \angle SUV = 180° - 120° = 60°$.
Since PW and QX are parallel, then $\triangle RUV$ and $\triangle RST$ have equal corresponding angles and are similar.
Therefore $\angle RVW = \angle VTX = 112°$.
Since UVW is a straight line, then $\angle RVU = 180° - \angle RVW = 180° - 112° = 68°$.
Since the measures of the angles in a triangle add to 180°:
$\angle URV + \angle RUV + \angle RVU = 180°$
$\angle URV = 180° - \angle RUV - \angle RVU = 180° - 60° - 68° = 52°$

P.10 Each diagram shows a triangle labelled with its area. Calculate the areas m, n and p.





We can solve this problem by constructing three right triangles around areas m,n, and p in every grid. This allows us to find the area of the full square grid and then subtract away the coloured areas made of right triangles. This will then leave us with just the areas of m,n, and p in each grid respectively.

The area of each grid= $4 \times 4 = 16$

We can find the sides of each of these coloured triangles by looking at coordinate positions.For example FB connects points $(4, 4)$ and $(1, 4)$. From looking at the x coordinate we can see that the length of FB is $4 - 1 = 3$ $units$. For line BE connecting $(4, 4)$ to $(4, 1)$ the difference is in the y coordinates. We can say that BE is $4 - 1 = 3$ $units$.

Repeating this we can find the side lengths of all the triangles. We can then calculated the area of our coloured regions.

In grid with area m, the area of the shaded region is divided into three triangles. The areas of the triangles can be found as follows:

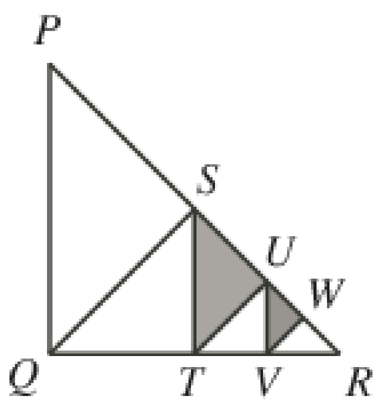
$$\triangle BEF : \ Area = \frac{3 \times 3}{2} = 4.5$$
$$\triangle AEO : \ Area = \frac{4 \times 1}{2} = 2$$
$$\triangle CFO : \ Area = \frac{4 \times 1}{2} = 2$$

$Area\ of\ shaded\ region = Area \triangle BEF + Area \triangle AEO + \triangle CFO$
$Area\ of\ shaded\ region = 4.5 + 2 + 2 = 8.5$

$Area\ m = Area\ grid - Area\ shaded$
$Area\ m = 16 - 8.5 = 7.5$

The same process can be repeated to find areas n and p.
$Area\ n = 16 - 6 - 1.6 - 2 = 6.5$
$Area\ p = 16 - 3 - 4 - 2 = 7$

P.11 In the right-angled $\triangle PQR$, $PQ = QR$. The segments QS, TU and VW are perpendicular to PR, and the segments ST and UV are perpendicular to QR, as shown. What fraction of $\triangle PQR$ is shaded?



Since $\triangle PQR$ is isosceles with $PQ = QR$ and $\angle PQR = 90°$, then $\angle QPR = \angle QRS = 45°$. Also in $\triangle PQR$, altitude QS bisects PR ($PS = SR$) forming two identical triangles, SQP and SQR.

Since these two triangles are identical, each has $\frac{1}{2}$ of the area of $\triangle PQR$.
In $\triangle SQR$, $\angle QSR = 90°$, $\angle QRS = 45°$, and so $\angle SQR = 45°$.
Thus, $\triangle SQR$ is also isosceles with SQ = SR.
Then similarly, altitude ST bisects QR ($QT = TR$) forming two identical triangles, SQT and SRT. Since these two triangles are identical, each has $\frac{1}{2}$ of the area of $\triangle SQR$ or $\frac{1}{4}$ of the area of $\triangle PQR$.
Continuing in this way, altitude TU divides $\triangle STR$ into two identical triangles, STU and RTU.
Each of these two triangles has $\frac{1}{4}$ of $\frac{1}{2}$ or $\frac{1}{8}$ of the area of $\triangle PQR$.
Continuing, altitude UV divides $\triangle RTU$ into two identical triangles, RUV and TUV.
Each of these two triangles has $\frac{1}{2}$ of $\frac{1}{8}$ or $\frac{1}{16}$ of the area of $\triangle PQR$. Finally, altitude VW divides $\triangle RUV$ into two identical triangles, UVW and RVW. Each of these two triangles has $\frac{1}{2}$ of $\frac{1}{16}$ or $\frac{1}{32}$ of the area of $\triangle PQR$. Since the area of STU is $\frac{1}{8}$ of the area of $\triangle PQR$, and the area of $\triangle UVW$ is $\frac{1}{32}$ of the area of $\triangle PQR$, then the total fraction of $\triangle PQR$ that is shaded is $\frac{1}{8} + \frac{1}{32} = \frac{1+4}{32}$ or $\frac{5}{32}$.