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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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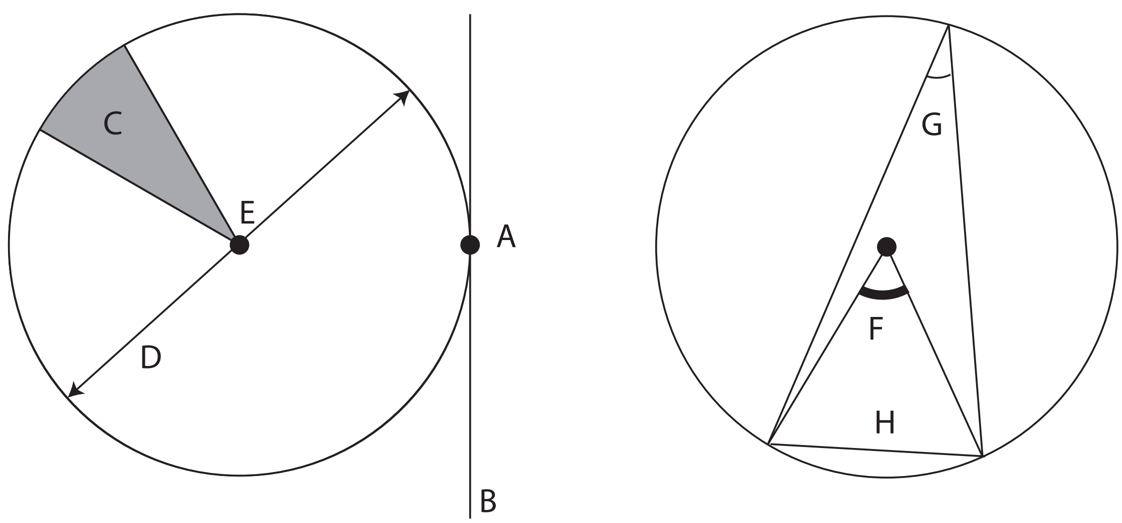
The Math Circles with Circles Solutions

Terminology Exercise:

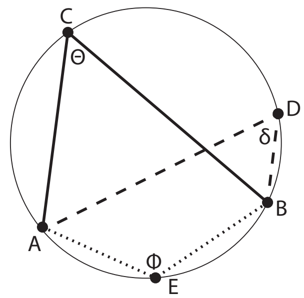
1. Label each letter with the correct name on the two circles below.

A: point, **B:** tangent, **C:** Sector, **D:** Diameter, **E:** Center, **F:** Central Angle, **G:** Inscribed Angle, **H:** Chord.



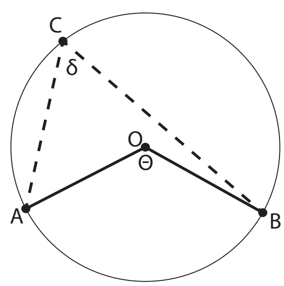
Circle Properties Practice:

1. If the angle θ on the diagram below is 37°, what is the angle δ? What is the angle ϕ?



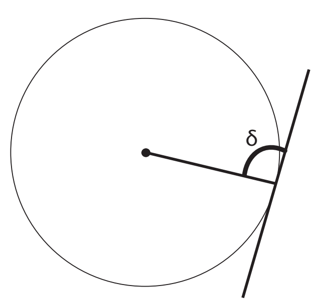
By ASAT $\delta = \theta = 37°$ since they are inscribed angles on the same chord.
By CQT opposite inscribed angles θ and ϕ add to 180°. Thus $\phi = 180° - 37° = 143°$

2. If the angle θ on the diagram below is 110°, what is the angle δ?



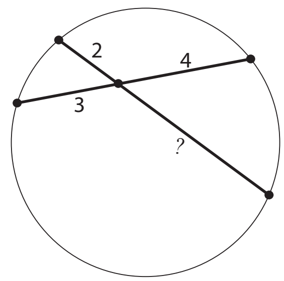
By STT the central angle θ is double the inscribed angle δ. Thus
$\delta = 110 \div 2 = 55$

3. What is the angle δ in the diagram below?



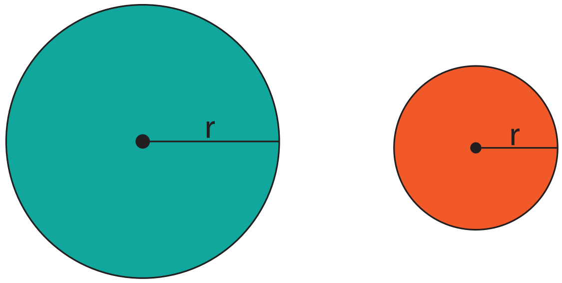
We have angle δ between a tangent line (only touches one point on the circle) and a radius. By TRT we can say the lines are perpendicular so $\delta = 90°$

4. What is the missing line segment length in the diagram below?



By CCT we say that $4 \times 3 = 2 \times ?$. We can rearrange this to find the missing side: If $2 \times ?$ is equal to 12 then ? must be 6.

Area of Circles: Let's look at the area of circles. Logically, the area of a circle depends on its radius. The larger the radius the larger the circle will be.



To be accurate the formula is $Area = \pi \times r^2$. or $Area = \pi(r \times r)$

Practice:

1. Find the area of a circle with radius 4cm. Try expressing your answer in terms of π.
$Area = \pi r^2$, r=4cm
$Area = \pi \times (4cm)^2$
$Area = 16\pi cm^2 \approx 50.27cm^2$ If we wanted to express area in terms of π we would stop at $Area = 16\pi cm^2$

2. What about the area of a circle with diameter 10cm? Radius is half the length of diameter.
$Radius = 10 \div 2 = 5$
$Area = \pi r^2$
$Area = \pi \times (5cm)^2$
$Area = 25\pi \approx 78.54cm^2$

Think...

3. Can we express the formula for area of a circle in terms of diameter? In terms of circumference?

Express radius in terms of diameter
$radius = \frac{diameter}{2}$ or $r = \frac{d}{2}$
$Area = \pi \times r^2$, can sub in $\frac{d}{2}$ for r
$Area = \pi \times (\frac{d}{2})^2$ Square top and bottom separate: $(\frac{d}{2})^2 = \frac{d^2}{2^2} = \frac{d^2}{4}$
$$Area = \frac{\pi \times d^2}{4}$$
Area is expressed in terms of diameter.

Express area in terms of Circumference:

$Circumference = 2 \times \pi \times radius$ or $C = 2\pi r$

By rearranging $r = \dfrac{C}{2\pi}$

Can substitute $\dfrac{C}{2\pi}$ in for r in area equation:

$Area = \pi r^2 \rightarrow Area = \pi(\frac{C}{2\pi})^2$

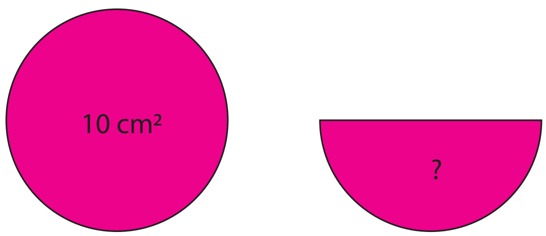
$(\frac{C}{2\pi})^2 = \frac{C^2}{4^2\pi^2}$

$Area = \dfrac{\pi \times C^2}{4\pi^2}$

$Area = \dfrac{C^2}{4\pi}$, $(\pi^2 \div \pi = \pi)$

Area is expressed in terms of Circumference.

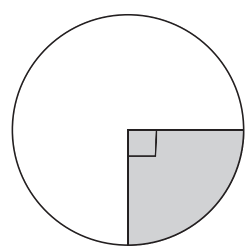
Area of a Sector: Given the following picture and told that the area of the circle on the left was $10cm^2$ what would you say that the area of the circle on the right is? $5cm^2$



We can say that the semi-circle on the right represents a sector of a circle. As mentioned before a sector is a section of a circle trapped by two radii like a piece of pi. All sectors can be described as a fraction of a circles total area.

We can see that the shaded sector on the circle to the right takes up a quarter of the circle. Thus we can say that:

$$Area\ of\ Sector = Area\ of\ Circle \times \frac{1}{4}$$

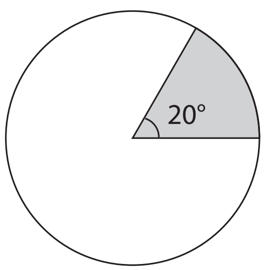


In general we can say:

$$Area\ of\ Sector = Area\ of\ Circle \times Fraction\ Shaded$$

What if it is not that easy to see the fraction? How can we find the area of the following shaded sector if the circle has radius 6 cm? What about the perimeter of the sector?

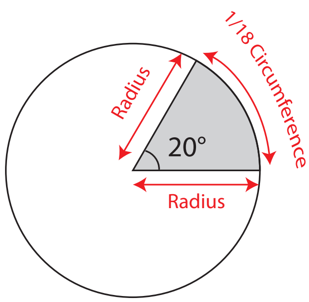
We can find the fraction of the circle that is shaded by comparing the central angle of the sector to the total degrees in a circle.



$$Fraction\ Shaded = \frac{Sector\ Angle}{Degrees\ in\ Circle}$$
$$Fraction\ Shaded = \frac{20^\circ}{360^\circ} = \frac{1}{18}$$

$Area\ of\ Sector = Area\ of\ Circle \times Fraction\ Shaded = (\pi 6^2) \times \frac{1}{18}$
$Area\ of\ Sector = 36\pi \times \frac{1}{18}$
$Area\ of\ Sector = 2\pi \approx 6.28cm.$

Finding Perimeter:



The arc at the edge of the sector can be said to be 1/18 of the circumference as we established that the sector central angle is 1/18 of the total circle.

The two edges in the circle are both radii as they run from the center to the edge of the circle. Thus we can say that:

$Perimeter = Radius + Radius + \frac{1}{18} \times Circumference$

$Perimeter = 2 \times Radius + \frac{1}{18} \times 2\pi r$

$Perimeter = 2 \times 6 + \frac{1}{18} \times 2 \times \pi \times 6$

$Perimeter = 12 + \frac{2 \times 6}{18}\pi$

$Perimeter = 12 + \frac{2}{3}\pi$

$Perimeter \approx 25.13cm$

How could the radius of a circle be found with just the area? For example what is the radius of a circle with $Area = 81\pi$.

$Area = \pi r^2$

Rearranging to solve for radius:

$r^2 = \frac{Area}{\pi}$, divide by π on both sides of the equation.

$\sqrt{r^2} = \sqrt{\frac{Area}{\pi}}$

$r = \sqrt{\frac{Area}{\pi}}, (r > 0)$

For a circle with area 81π:

$r = \sqrt{\frac{81\pi}{\pi}}$

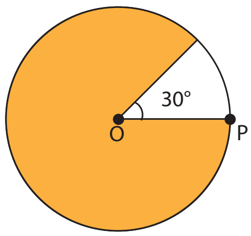
$r = \sqrt{81}$

$r = 9(r > 0)$

The tools learned today provide an excellent introduction to a world of geometrical problems. The following problems will require you to apply several of these tools together. Some questions are challenging and will require a bit of thinking so don't be discouraged if you do not arrive at an answer immediately! Remember some of the tools we used last week!

PROBLEM SET:

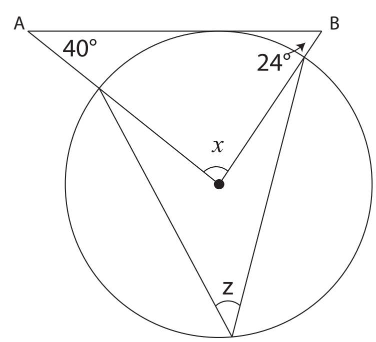
1. The length of OP is 3cm what is the area of the **white** section of the circle.



The shaded sector on the circle has a central angle of $360° - 30° = 330°$
We can find the fraction that is shaded by comparing 330° to the total degrees in the circle 360°.
$Fraction\ White = \frac{30°}{360°} = \frac{1}{12}$
The area of the full circle can be found by $area = \pi r^2$, where r is the radius of the circle. Since length OP runs from the center to the circumference of the circle we can call it a radius. $OP = Radius = 3cm$.
Thus $Area = \pi \times 3cm^2$
$Area = 9\pi$

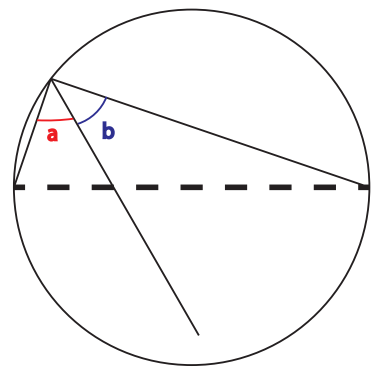
$Area\ Sector = Area\ total\ Circle \times Fraction\ White$
$Area\ Sector = 9\pi \times \frac{1}{12}$
$Area\ Sector = \frac{9}{12}\pi = \frac{3}{4}\pi \approx 2.36cm^2$

2. Find angles x and z. $\angle$ B is 24°.



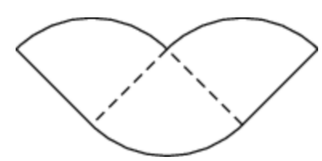
From the sum of angles in a triangle we can say that $40° + 24° + x = 180°$ using the triangle made with vertices A,B, and the center of the circle.
Rearranging to solve for x:
$x = 180° - 24° - 40° = 116°$
We can then find $\angle z$ using STT:
By STT z is half the angle of x: $z = x \div 2$
$z = 116 \div 2 = 58$

3. In the diagram below what is the relationship between angle a and angle b? If $b = 27°$ what is a?



By ASAT angles a and b add up to 90° because they are inscribed on a diameter. Thus we can say the angles are complementary.
Mathematically $\angle a + \angle b = 90°$, also we know $b = 27°$
$\angle a + \angle 27° = 90°$
$\angle a = \angle 90° - \angle 27°$
$\angle a = \angle 63°$

4. A circle with area 36π is cut into quarters and three of the pieces are arranged as shown. What is the perimeter and area of the resulting figure?



To find the Area:
We have 3 quarter circles pieced together in the figure shown.
Thus we have $\frac{3}{4}$ of the original area of the circle left.
$Area\ of\ Figure = Area\ of\ Full\ Circle \times \frac{3}{4}$
$Area\ of\ Figure = 36\pi \times \frac{3}{4}$
$Area\ of\ Figure = 27\pi$

To Find the Perimeter:

We have 5 discrete sides to the perimeter. We have 3 quarter arcs and 2 radii.

The quarter arcs are each a quarter of the original circles circumference.

Thus $Perimeter = \frac{3}{4} \times circumference + 2 \times Radius$

We must find the radius of the original circle:

Using the result from the end of the lesson, we can say that $radius = \sqrt{\frac{Area}{\pi}}$

$radius = \sqrt{\frac{36\pi}{\pi}} = \sqrt{36} = 6$

Finding Circumference:

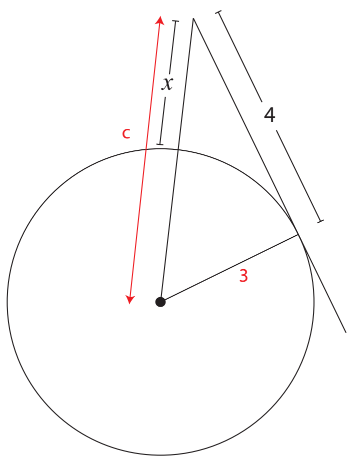
$Circumference = \pi \times d = 2\pi r$

$Circumference = 2\pi \times 6$

$Circumference = 12\pi$

$Perimeter = \frac{3}{4} \times 12\pi + 2 \times 6$ $Perimeter = 9\pi + 12 \approx 49.7$

5. Given that the circle has radius 3 what is the length of x?



The bottom side of the triangle in the figure is also the radius of the circle. Thus we can say that the bottom side is length 3.

We can find the longest side which we call c with Pythagorean theorem. $3^2 + 4^2 = c^2$

$c = \sqrt{3^2 + 4^2} = 5$

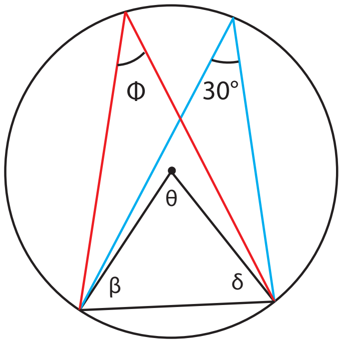
We can see that c is as long as x and the radius of the circle combined:

$c = x + 3$

$5 = x + 3$

$x = 2$

6. Find the measure of angles β , θ, δ and ϕ.



By ASAT we can say that angles ϕ and $30°$ must be the same as they are inscribed angles on the same chord.Thus $\phi = 30°$

By STT we can say that angle θ is twice as large as ϕ and $30°$. Thus $\theta = 2 \times 30° = 60°$. β,δ and θ are 3 angles that make up a triangle. The top two sides are both radii of the circle as they run from the circumference to the center. Thus we can say that the two sides are equal so angles β and δ must be equal.

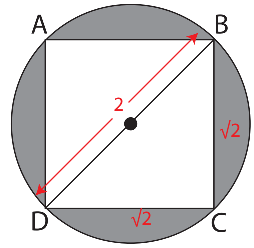
If $\theta = 60°$ and $\beta = \delta$, using sum of angles in a triangle:

$60° + \beta + \delta = 180°$

$\beta + \delta = 120°$

If β and δ are equal are both equal they must both be half of $120°$. Thus $\beta = \delta = 60°$.

7. The square $ABCD$ is enclosed in a circle. Find the area of the shaded region if the square has side lengths of $\sqrt{2}$.



The area of the shaded region is equal to the area of the entire enclosing circle minus the area of the square $ABCD$. The square has side lengths of $\sqrt{2}$ so the area can be found:
$Area\ of\ Square = \sqrt{2} \times \sqrt{2} = 2$

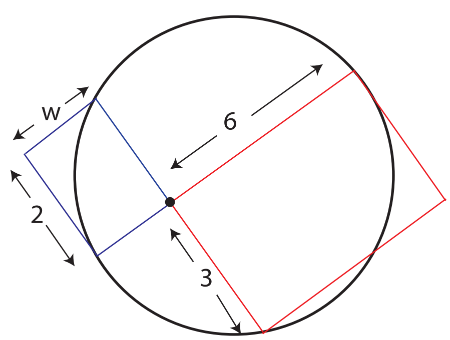
We can find the radius of the circle by forming a triangle with the sides of the square. Using pythagorean theorem to find DB in the figure above as the diagonals of a square are perpindicular. (Meaning the central angle is 90°)
$DB^2 = \sqrt{2}^2 + \sqrt{2}^2 = 4$
$DB^2 = 4$
$DB = 2$

DB measures the diameter of the circle. To find the diameter we can divide DB by 2:
$Radius = 2 \div 2 = 1$

$Area\ of\ Circle : \pi \times r^2 = \pi \times 1^2 = \pi$
$Area\ of\ Square : \sqrt{2} \times \sqrt{2} = 2$

$Area\ Shaded = Area\ Circle - Area\ Square$
$Area\ Shaded = \pi - 2 \approx 1.14$

8. Find the ratio between the areas of the two rectangles. (Refer to when we found the ratio of circumference and diameter in the warm-up.)

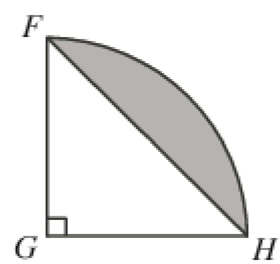


The sides of the rectangles inside the circle are crossing chords. Thus we can say by CCT:
$6 \times w = 3 \times 2$
$6w = 6$

$w = 1$

Area of smaller rectangle $= 2 \times 1 = 2$
Area of larger rectangle $= 6 \times 3 = 18$
Ratio of Areas: $\frac{18}{2} = 9$ or $2 : 18 \rightarrow 1 : 9$
The larger rectangle is 9 times larger than the smaller rectangle.

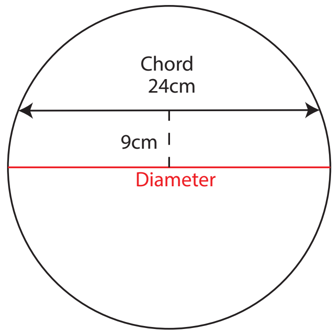
9. In right-angled, isosceles triangle FGH, FG=$\sqrt{8}$. Arc FH is part of the circumference of a circle with centre G and radius GH, as shown. What is the area of the shaded region?



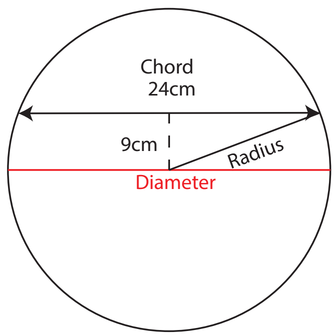
Arc FH is a quarter circumference of a circle centered at G. Thus we can say the circle has radius $FG = GH = \sqrt{8}$ Notice how the figure shows a quarter with a triangle FGH cut out of it.
Thus to find the shaded area we can find the difference of the areas of the quarter circle and $\triangle FGH$.
$Shaded\ Area = Area\ Quarter\ Circle - Area \triangle\ FGH$
$Area\ of \triangle\ FGH = \dfrac{base \times height}{2} = \dfrac{\sqrt{8} \times \sqrt{8}}{2} = \dfrac{8}{2} = 4$
$Area\ Quarter\ Circle = Area\ Circle \times \dfrac{1}{4}$
$Area\ Quarter\ Circle = \pi(\sqrt{8})^2 \times \frac{1}{4} = 8\pi \times \frac{1}{4} = 2\pi$

$Shaded\ Area = 8\pi - 4 \approx 21.13$

10. A 24cm chord is placed on a circle parallel to the diameter. The distance between the chord and the diamater is 9cm. Find the diameter of the circle in cm. *Hint: Draw a Diagram! Diamater is double the Radius!*



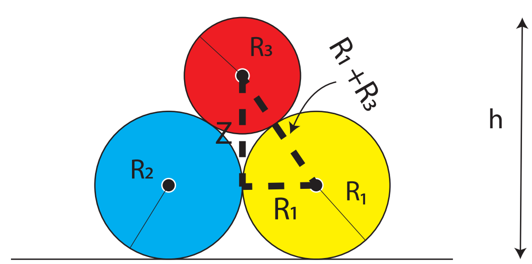
We can arrive at this diagram from the information given.
We can find the radius by making a triangle with half the chord and the height $9cm$.



Using Pythagorean Theorem:
$Height^2 + 1/2Chord^2 = Radius^2$
$9cm^2 + 12cm^2 = Radius^2$
$225 = Radius^2$
$25cm = Radius$

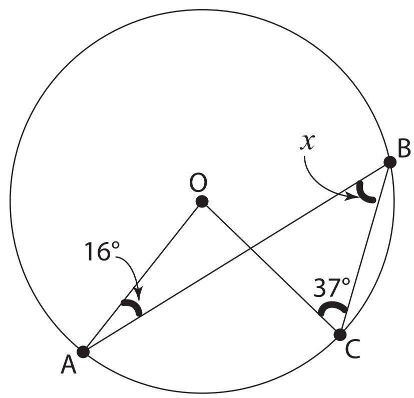
The Diameter is double the Radius: $Diameter = 2 \times 25cm = 50cm$

*11. Find the height of the stack of balls in terms of the radii of the three balls. **The blue and yellow balls are the same size.**



We can say that $h = R1 + Z + R3$ (Note $R1 = R2$)
We can find Z using Pythagorean theorem:
$Z^2 + R1^2 = (R1 + R3)^2$
We can solve for the missing side Z by rearranging:
$Z = \sqrt{(R1 + R3)^2 - R1^2}$
Thus we can say the height of the balls h is:
$h = R1 + Z + R3$
$h = R1 + \sqrt{(R1 + R3)^2 - R1^2} + R3$, can expand out $(R1 + R3)^2 = (R1 + R3)(R1 + R3) = R1^2 + 2R1R3 + R3^2$
$h = R1 + \sqrt{R1^2 + 2R1R3 + R3^2 - R1^2} + R3$
$h = R1 + \sqrt{2R1R3 + R3^2} + R3$

*12. Find the missing angle x below.
Hint: Recall what was shown in question in 5.



While it may not be visually obvious, we can use STT, as B is on the major arc of AC. By STT, angle AOC is $2x$. Using the hint, we can label the unlabelled angle in both triangles as y. This gives us $180 = 2x + 16 + y$ and $180 = x + 37 + y$. Since both equations are equal to 180, they are equal to each other. So we have $2x + 16 + y = x + 37 + y$. We begin to "balance" this equation by subtracting y from both sides. This gives us $2x + 16 = x + 37$. We now subtract x from both sides, resulting in $x + 16 = 37$. We finish off balancing our equation by subtracting 16 from both sides, resulting in our final answer, which is $x = 21°$.