UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
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Grade 7/8 Math Circles

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Probability Solutions

Excercises:

1. You roll a six-sided die a single time.

   (a) What is the probability of rolling a 2?

   1 possible way to roll a 2, 6 total possible outcomes when rolling a die. $Probability = \frac{1}{6}$.

   (b) What is the probability of rolling a number that is even?

   3 possible even outcomes {2,4,6}, 6 total possible outcomes when rolling a die. $Probability = \frac{3}{6} = \frac{1}{2}$.

   (c) What is the probability of rolling a prime number?

   ***Hint**: A **prime** is an integer greater than 1 such that its only positive divisors are 1 and itself (e.g. 2, 3, 5, 7, 11, ...).*

   3 possible prime outcomes {2,3,5}, 6 total possible outcomes when rolling a die. $Probability = \frac{3}{6} = \frac{1}{2}$.

   (d) What is the probability of rolling a 7?

   There are 0 ways to roll a 7, and 6 posssible outcomes. $Probability = \frac{0}{6} = 0$

2. You draw a single card from a deck of 52 cards.

   (a) What is the probability to draw the 2 of spades?

   When drawing a card from a standard deck there is 1 possible 2 of spades card. There are 52 total possible cards to draw. Therefore my probability is $\frac{1}{52}$.

   (b) What is the probability to draw a diamond?

   There are 13 diamonds in a deck of cards. $Probability = \frac{13}{52} = \frac{1}{4}$

(c) What is the probability to draw an Ace?

There are 4 Aces in a deck of cards. $Probability = \frac{4}{52} = \frac{1}{13}$

3. There are 6 beads in a bag, 3 are red, 2 are yellow and 1 is blue. What is the probability of picking a yellow? What is the sample space of the bag? The sample space is the total possible outcomes when drawing from the bag. When I draw my bead it could be any of {red,yellow,blue}. There are 2 yellow beads and 6 total beads. $Probability = \frac{2}{6} = \frac{1}{3}$

Relationships

Exercises:

1. State whether the following pairs of events are Mutually-Exclusive, Independent or Dependent:

   (a) Drawing a 7 from a deck of cards AND drawing a Jack after without replacement. Dependent

   (b) Drawing a 7 AND an 8 from a deck in one draw. Mutually-Exclusive

   (c) Getting a head on a coin-toss AND getting a tail on a different coin-toss. Independent

   (d) Flipping a coin AND rolling a 6 on a die. Independent

Intersection and Union

Intersection

For two events A and B, the ***intersection*** *of A and B, denoted* $(A \cap B)$*, is the event that both A* ***AND*** *B happen.*

Exercise: You roll a 6-sided die. Let event A be that the number rolled is greater than 2. Let event B be that the number rolled is odd. What is the **intersection**, $(A \cap B)$ of the two events?

First we finds the outcomes that satisfy event A. All numbers we can roll that are greater than 2 can be found as $A : \{3, 4, 5, 6\}$

For the outcomes that satisfy event B we have all the odd numbers we could roll. $B : \{1, 3, 5\}$

The **intersection**, $(A \cap B)$ is all outcomes that satisfy event A and B. Numbers that are odd and greater than 2. These numbers should be in both our sets A and B.

Thus, our interesection from looking at sets A and B can be found as: $(A \cap B) = \{3, 5\}$

Union

*For two events A and B, the **union** of events A and B, denoted $(A \cup B)$, is the event that A **OR** B occurs.*

Exercise: You draw a card from a standard deck. Let event A be that the card is a heart. Let event B be that the card is a Queen. What is the **union**, $(A \cup B)$ of the two events?

Event A is that the card is a heart. The possible outcomes for event A to happen are just one of the 13 hearts in the deck. A: {2,3,4,5,6,7,8,9,10,J,Q,K,A}. Event B is that the card is a queen. The possible outcomes for event B to happen are the 4 queens in the deck. B: {Q hearts, Q diamonds, Q spades, Q clubs}

To find the total possible outcomes where A or B occur we add our choices for A and B together. We must be careful though, note that the Queen of Hearts is counted on both of the sets. I can only draw the Queen of hearts once so I should avoid double counting it as two possible outcomes. $(A \cup B)$ = (13 hearts) + (4 Queens) - (Queen of hearts). The union of the two events is then {2 hearts, 3 hearts, 4 hearts, 5 hearts, 6 hearts, 7 hearts, 8 hearts, 9 hearts, 10 hearts, Jack hearts, Queen hearts, King hearts, Ace hearts, Queen clubs, Queen spades, Queen diamonds} or 16 possible outcomes rather than 17.

Practice: If events A and B are mututally exclusive what is the intersection $(A \cap B)$ of the two events?

If events are mututally exclusive they can never happen at the same time. As interesection is the event that both events happen, the intersection is empty as no outcomes allow A AND B to occur.

Compund Probabilities: Multiple Events

Sum Rule

Example: Given a bag of 3 blue balls, 5 red balls, 6 yellow balls and 6 green balls, what is the probability of picking a red ball or a green ball?

Notice how you cannot pick a ball that is both red and green so the two events are mutually exclusive so we can apply the **Special Sum Rule** as follows:

$$\text{P(green or red ball)} = \text{P(green ball)} + \text{P(red ball)} = \frac{6}{20} + \frac{5}{20} = \frac{11}{20} = 0.55$$

Example: The numbers 1 to 20 are written on a paper and placed in a bag. What is the probability of picking a number divisible by 2 or a number divisible by 3?

Let event A be that you pick a number divisible by 2. Let event B be that you pick a number divisible by 3. Notice that if you pick the number 6 then since 6 is divisible by 2 and by 3 and so $A \cap B$ contains at least 6. So we must apply the **General Sum Rule** as follows:

$$\text{P(divisible by 2 or 3)} = \text{P(divisible by 2)} + \text{P(divisible by 3)} - \text{P(divisibile by 2 and 3)}$$
$$\text{P(divisible by 2 or 3)} = \frac{10}{20} + \frac{6}{20} - \frac{3}{20} = \frac{13}{20} = 0.65$$

Note: *There are 10 numbers divisible by 2, 6 divisble by 3 and 3 divisible by both 2 and 3.*

Compund Probabilities: Multiple Events

Exercises

1. **Dependent**: A table of 5 students has 3 seniors and 2 juniors. The teacher is going to pick 2 students one after another at random from this group to present homework solutions. Find the probability that both students selected are juniors.

   Teacher picks a Junior the first time AND a Junior the second time. The "AND" indicates we use product rule.

   Let event A be the teacher picking a Junior the first time. Let event B be the teacher picking a Junior the second time.

   Event A: 2 Juniors, 5 students. $\text{P(A)} = \frac{2}{5}$

Event B: Now after the teacher picks a Junior, there is only 1 Junior and 4 students left. P(B)= $\frac{1}{4}$

Product rule tells us the probability A and B occur $(A \cap B)$ = P(A) × P(B)

$(A \cap B)$= $\frac{2}{5} \times \frac{1}{4} = \frac{2}{20} = \frac{1}{10}$ or 0.1 probability the teacher picks two juniors.

2. You draw a card from a deck of 52 cards.

   (a) What is the probability of drawing a 5 or a spade?

   OR → Sum rule

   Let event A be drawing a 5, let event B be drawing a spade.

   P(A) = $\frac{4}{52}$, P(B) = $\frac{13}{52}$

   We have to be careful because A or B are not mutually exclusive. If I draw a 5 of spades I perform A and B. Probability of drawing the 5 of spades: P$(A \cap B)$ = $\frac{1}{52}$

   By sum rule our probability for A or B to occur is: P$(A \cup B)$ = P(A) + P(B) - P$(A \cap B)$

   P$(A \cup B)$ = $\frac{4}{52} + \frac{13}{52}$ - $\frac{1}{52}$

   P$(A \cup B)$ = $\frac{16}{52} = \frac{4}{13}$ chance to draw a 5 or a spade.

   (b) What about the probability of drawing a heart or a spade?

   OR → Sum rule

   These events are mutually exclusive as I can never draw a heart or spade. Therefore there is no intersection and we can use Special Sum Rule.

   Let event A be drawing a heart, event B be drawing a spade.

   They have the same probability: P(A)=P(B)=$\frac{13}{52}$

   The probability of A or B happening: P$(A \cup B)$ = P(A) + P(B)

   P$(A \cup B)$ = $\frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$ chance to draw a heart or spade.

   (c) Find the probability of drawing two cards one after another and having them both be hearts (without replacement).

   Let event A be drawing a heart, let event B to draw another heart. We use product rule as we are drawing a heart AND another heart after. P(A) = $\frac{13}{52} = \frac{1}{4}$. After drawing the first heart there is one less heart in the deck but also one less card in the deck.

We have 3 hearts and 51 total cards for the second draw. Therefore P(B) $= \frac{12}{51}$.

By product rule the probabilty of A and B : P($A \cap B$)= P(A) $\times$ P(B)

P($A \cup B$) $= \frac{1}{4} \times \frac{12}{51} = \frac{12}{202} = \frac{1}{17}$ chance to draw two hearts in a rwo without replacement.

3. You roll two 6-sided dice.

   (a) Find the number of possible outcomes in the sample space.

   (b) What is the probability that both die show 1?

   (c) What is the probability that the sum of the two dice is an odd number that is greater than 5?

   (d) What is the most likely sum to occur from the roll?

a) The first thing we always want to do is find the number of possible outcomes in the sample space. Since there are 6 possible outcomes for the first die, and for each of those outcomes we have 6 more when we roll the second die, in our sample space there are:

$$6 \times 6 = 36 \text{ possible outcomes.}$$

A good way to visualize why this is true is listing the outcomes in the shape of a square:

Die 1

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 2	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

b) We can see from the square that there is only one possible outcome to roll a 1 on both die. Therefore our probability to roll a 1 on both die is $\frac{1}{36}$. Also using product rule where A and B are rolling a 1: $\text{P}(A \cap B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

c) The odd sums greater than 5 are 7, 9 and 11.
From the table we can see that 7 appears 6 times from the combination (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3).
Similarly, 9 appears 4 times from the combination (3,6), (6,3), (4,5), and (5,4) and 11 appears 2 times from the combination (5,6) and (6,5).
And so the probability of getting an odd sum greater than 5 is as follows:

$$\text{P(odd sum greater than 5)} = \text{P(7 or 9 or 11)} = \text{P(7)} + \text{P(9)} + \text{P(11)} \ \textit{(by the sum rule)}$$

$$\text{P}(7) + \text{P}(9) + \text{P}(11) = \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{12}{36} \approx 33\%$$

And so the probability of rolling an odd sum greater than 5 is approximately **33%**.

d) We can see from the table above that 7 is the most likely sum to be rolled with 6 possible outcomes. ($\text{P}(7) = \frac{6}{36} = \frac{1}{6}$)

Complement

Examples:

1. What is the probability of spinning the above spinner and it landing on yellow? What is the complement of this proability?

   Assuming each section of the spinner is equal in size, There is a $\frac{1}{3}$ chance to spin yellow.
   If event A is spinning yellow: $\text{P(A)} = \frac{1}{3}$
   The complement can be found as: $\text{P}(\bar{A}) = 1 - P(A)$

$$P(\bar{A}) = 1 - \frac{1}{3} = \frac{2}{3}$$

Sometimes it can be useful when finding the probabilities of certain events to find the probability of their complements instead. We can also rearrange the complement equation as $P(A) = 1 - P(\bar{A})$.

2. A card is chosen at random from a deck of cards. What is the probability that the card is not a Jack? P(Not Jack) = 1 - P(Jack)
   P(Jack) $= \frac{4}{52} = \frac{1}{13}$
   P(Not Jack) = 1 - P(Jack) = 1 - $\frac{1}{13} = \frac{12}{13}$

Conditional Probability

Examples:

Find the Probability of event B given that event A occurs and also has a 0.45 (45%) probability to occur. The probability of event A and event B occuring is 0.2 (20%.)

We can use the formula $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$
P(A) = 0.45, P($B \cap A$)=0.2
$P(B \mid A) = \frac{0.2}{0.45} \approx 0.44$ *or* 44%

In a group of 100 pet owners, 40 own cats, 30 own dogs, and 20 own a dog and a cat. If a pet owner chosen at random owns a cat, what is the probability they also own a dog? What is the probability they do not own a dog?

Let event A be to own a cat, event B to own a dog.
P(cat)=P(A)=$\frac{40}{100} = 0.4$
P(cat and dog) = P($A \cap B$) = $\frac{20}{100} = 0.2$
$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.2}{0.4}$= 0.5 chance to own a dog given owning a cat.
Using the complement P($\bar{A}$) = 1 - P(A) = 1 - 0.5 = 0.5 The complement event to owning a dog is not owning a dog as shown above so the two probabilities should add to 1. Thus the probability that a cat owner doesnt own a dog is also 0.5.

Problem Set:

1. Find the following probabilities

   (a) What is the probability of getting a head or tail when flipping a coin? $\frac{1}{2}$

   (b) What is the probability of rolling a 7 on a 6-sided die? 0

   (c) What is the probability of drawing a Jack from a deck of cards? $\frac{1}{52}$

   (d) The probability that there is a Math Circles lesson today is 3/7. What is the probability that there is not a Math Circles lesson? $\frac{4}{7}$

   (e) You time travel to a random day in the next leap year. What is the probability that the day is Christmas? $\frac{1}{366}$

2. Identify the following pairs of events as independent, dependent, or mutually exclusive.

   (a) Driving fast and getting a speeding ticket. Dependent

   (b) Picking a diamond from a deck of cards and then drawing another diamond without replacing the first card. Dependent

   (c) Picking a diamond and then picking a spade from a deck of cards (without replacement).Dependent

   (d) Flipping a coin and getting heads and then drawing the 3 of spades form a deck of cards. Independent

   (e) A student being in math circles and having interest in math. Dependent

   (f) Flipping a coin and getting heads and tails. Mutually-Exclusive

   (g) Picking two red balls consecutively (back to back) when picking from a bag containing 10 red, 12 black and 2 white balls (with replacement). Independent

3. Two 6 sided die are rolled. What is the probability that:

Die 1

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 2	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(a) their sum is a prime number?

Looking at all the possible sums, we note the prime sums are 2, 3, 5, 7, and 11. So we are looking at the outcomes that give us a sum of 2 or 3 or 5 or 7 or 11. These events are mutually exclusive as you cannot get a sum that is 5 and 11 or a sum that is 3 and 11 and so on. So we can apply the Special Sum Rule as follows:

$$\text{P(odd prime)} = \text{P}(2) + \text{P}(3) + \text{P}(5) + \text{P}(7) + \text{P}(11)$$

$$\text{P(odd prime)} = \frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36} \approx 0.42 = 42\%$$

(b) their sum is not a prime number?

One way to solve this is to count all the possible sums that are not prime and use Sum Rule to find the total probability.

However, we know that number is either prime or it isn't prime (the event of being prime and the event of not being prime are mutually exclusive). So of all the possible sums, each sum is either prime or it isn't prime. Since we know that 42% of the possible sums are prime, then we can find the sums that are not prime as follows:

$$\text{P(all possible sums) - P(prime sums)} = \text{P(sums that are not prime)}$$

$$\text{P(sums that are not prime)} = 1 - \frac{15}{36} = \frac{21}{36} \approx 0.58 = 58\%$$

(c) they both show the same number? For this to happen, both die need to be 1 or both be 2 or both be 3, ..., or both be 6. These 6 events are all mutually exclusive so we can apply the Special Sum Rule as follows:

$$\text{P(both show the same number)} = \text{P(both 1)} + \text{P(both 2)} + \text{P(both 3)} + \text{P(both 4)} + \text{P(both 5)} + \text{P(both 6)}$$

Now the probability of both die showing the number 1 means that die 1 needs to be 1 **and** die 2 needs to be 1. This is the Product Rule as follows:

$$\text{P(both show 1)} = \text{P(die 1 is 1)} \times \text{P(die 2 is 1)} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Applying this to the other possibilities 2, 3, 4, 5, and 6, we get:

$$\text{P(both show the same number)} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} \approx 0.17 = 17\%$$

Hint: *Use the chart showing the possible outcomes when rolling two die for part (a) and (b).*

4. You have a bag with 13 purple and 24 yellow marbles. 5 of the purple marbles are large and the rest are small. 9 of the yellow marbles are large and the rest are small. What is the probability of each of the following:

   (a) picking a yellow marble?

   There are 13 purple marbles and 24 yellow marbles so there are 37 marbles in total.

   $$P(yellow\ \ marble) = \frac{24}{37}$$

   (b) picking a small marble?

   There are 8 small purple marbles and 15 small yellow marbles so there are 23 small marbles in total.

   $$P(small\ \ marble) = \frac{23}{37}$$

(c) picking a purple and small marble?

There are 13 purple marbles and 8 of them are small.

$$P(purple\ \ and\ \ small\ \ marble) = \frac{8}{37}$$

(d) picking a yellow or large marble?

There are 24 yellow marbles and 14 large marbles and 9 marbles that are both yellow and large. So we apply the General Sum Rule to get:

$$P(yellow\ \ or\ \ large\ \ marble) = \frac{24}{37} + \frac{14}{37} - \frac{9}{37} = \frac{29}{37}$$

5. Find the following Probabilities:

(a) What is the probability that I roll a 3 on a die twice in a row?

Say event A is to roll a 3 on the first roll. Event B is to roll a 3 on the second roll. The events are independent and will have equal probability. We can use product rule to find the probability of A AND B occuring. P(A)=P(B)=$\frac{1}{6}$.

Probability of A and B: $P(A \cap B) = P(A) \times P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

(b) What is the probability that I draw a Queen of Hearts twice in a row (I replace the first Queen and draw again)?

Say event A is to draw a queen of hearts on the first draw. Event B is to draw a queen of hearts on the second draw. Since we replace the card we can say the two events are independent. I draw a queen of hearts from a deck of 52 cards each time. Thus each event has a probability of $\frac{1}{52}$. To find the probability of A AND B we can use product rule: $P(A \cap B) = P(A) \times P(B) = \frac{1}{52} \times \frac{1}{52} = \frac{1}{2704}$ chance to draw the queen of hearts twice in a row.

6. What is the probability of **winning** (ties do not count as winning) a game of Rock, Paper, Scissors? (assume each of rock, paper, and scissors is just as likely to be chosen.)

(a) What is the probability to not win a game of rock paper scissors? When my opponent picks rock, paper, or scissors I have the ability to make 3 different selections. One where I win, one where I tie, one where I lose. For example if my opponent picks rock there

are 3 outcomes: **Win:** I pick paper , **Tie:** I pick rock , **Lose:** I pick scissors.
I have an equal chance to pick each option meaning I also have an equal chance of $\frac{1}{3}$ to win, tie, or lose.
Thus I have a Probability of $\frac{1}{3}$ to win, and a probability of $\frac{2}{3}$ to not win (tie or lose).

(b) Your friend argues that you are more likely to get heads twice in a row on two consecutive coin flips than not win three games of rock paper scissors in a row. Is your friend correct?
The probability to not win multiple games can be found with product rule (not win AND not win AND not win). Each individual game you have a $\frac{2}{3}$ chance to not win. Say I have 3 events A, B, and C which are all to not win. The product rule extends to 3 events and the probability can be found as: P($A \cap B \cap C$) = P(A) × P(B) × P(C) = $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$ chance to lose 3 games in a row.
The probability to get heads on two different flips can also be found with product rule (heads AND heads). If events D and E are both getting heads on each coinflip. The probability of A and B : P(D ∩ E) = P(D) × P(E) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ chance to flip heads twice.
Comparing the probabilites: The chance to lose 3 Rock, Paper, Scissors games is $\frac{8}{27} \approx$ 0.3 or 30%. The chance to flip heads twice in a row is $\frac{1}{4} = 0.25$ or 25%. Thus you are more likely to lose 3 games of rock, paper, scissors than flip heads twice. Your friend is wrong.

7. A weighted coin *(it is no longer fair)* is altered so that the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability that head appears on the first flip and tail appears on the last flip?
First, since P(head) = $\frac{5}{7}$ then we must have $P(tail) = \frac{2}{7}$.
We want the first flip to be heads and the last to be tails but the second flip is not specified so it can be a head **or** a tail so we use Sum Rule to get P(HHT or HTT) = P(HHT) + P(HTT).
We must find P(HHT) and P(HTT) using Product Rule as follows:

$$P(HHT) + P(HTT) = P(H) \times P(H) \times P(T) + P(H) \times P(T) \times P(T)$$

$$P(HHT or HTT) = \frac{50}{343} + \frac{20}{343} = \frac{70}{343} \approx 0.20 = 20\%$$

8. The Ministry of Magic is holding a lottery and has sold 2000 tickets. If Harry Potter has a $\frac{1}{16}$ chance of winning, then how many tickets did he purchase?

$$P(winning) = \frac{Number\ of\ tickets}{Total\ number\ of\ tickets} = \frac{Number\ of\ tickets}{2000} = \frac{1}{16}$$

We can use $16 \times 125 = 2000$ and the equality of fractions to find that he bought 125 tickets.

9. Answer the following questions using conditional probability:

(a) The probability that it is Monday and that a student is absent is 0.12. Since there are 5 school days in a week, the probability that it is Monday is 0.2. What is the probability that a student is absent given that today is Monday?

Call event A the event where today is friday. Call event B that a student is absent. We want to find the probability that a student is absent given that today is Monday. We find this with the formula: $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$

- P(B | A) is the probability that a student is absent given it is Monday.
- The probability that today is Monday and a student is absent : P(A ∩ B) = 0.12
- The probability that today is Monday during a school week is $\frac{1}{5}$ or 0.2

Using the formula : $\text{P(Absent} \mid \text{Student)} = \dfrac{\text{Monday and Absent}}{\text{Monday}} = \dfrac{0.12}{0.2} = 0.6$

Thus there is a probability of $\frac{6}{10}$ or a 60% chance that a student is absent given that is Monday.

(b) Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

If A and B are mutually-exclusive then the intersection, $P(A \cap B) = 0$ since event A AND event B can never occur at the same time. Therefore when plugging $P(A \cap B) = 0$ into our equation for conditional probability we get that $P(A \mid B) = 0$.

(c) In your math class, 30% of the students passed both tests on the probability unit and 45% of them passed the first test. What percent of students that passed the first test also passed the second one?

Let A be the event that they passed the first test and B be the event that they passed the second test. We are given that they passed the first test which means that P(A) = 45%. We now want to know that given A, what was the probability of B and so we use conditional probability:

$$P(pass\ the\ second\ test\ given\ that\ they\ passed\ the\ first) = \frac{P(passed\ both)}{P(passed\ the\ first)}$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.30}{0.45} \approx 0.67 = 67\%$$

(d) If P(A) = 10%, P(B) = 45%, and $P(A \cup B) = 50\%$, find $P(A \mid B)$. *Hint: Look at Special Sum Rule from Before!*

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so 50% = 10% + 45% - $P(A \cap B)$ which gives $P(A \cap B) = 5\%$.

Then $P(A \mid B) = \frac{5\%}{45\%} \approx 11\%$.

10. Ms. Ganji is checking for homework completion. Each student has a 60% chance of having completed their homework. Ms. Ganji selects two students at random for homework check. What is the probability that:

(a) both students have completed their homework?

P(student has completed their homework) = 60%

P(student has not completed their homework) = 100% - 60% = 40%

$$\begin{aligned} P(\text{Both completed homework}) &= P(\text{Student 1 complete} \cap \text{Student 2 complete}) \\ &= P(\text{Student 1 complete}) \times P(\text{Student 1 complete}) \\ &= 0.6 \times 0.6 \\ &= 0.36 \\ &= 36\% \end{aligned}$$

(b) neither student has completed their homework?

$$\begin{aligned} P(\text{Both incomplete homework}) &= P(\text{Student 1 incomplete} \cap \text{Student 2 incomplete}) \\ &= P(\text{Student 1 incomplete}) \times P(\text{Student 1 incomplete}) \\ &= 0.4 \times 0.4 \\ &= 0.16 \\ &= 16\% \end{aligned}$$

(c) only one student has completed their homework?

$$\begin{aligned} P(\text{Only completed homework}) &= P(\text{Only student 1 complete} \cup \text{Only student 2 complete}) \\ &= P(\text{Only student 1 complete}) + P(\text{Only student 1 complete}) \\ &= P(\text{Student 1 complete}) \cap P(\text{Student 2 incomplete}) \\ &\quad + P(\text{Student 1 incomplete}) \cap (\text{Student 2 complete}) \\ &= (0.6 \times 0.4) + (0.4 \times 0.6) \\ &= 0.48 \\ &= 48\% \end{aligned}$$

11. * Alice rolls a standard 6-sided die. Bob rolls a second standard 6-sided die. Alice wins if the values shown differ by 1. What is the probability that Alice wins? (Source: 2009 Pascal (Grade 9), #21)

We can use our chart from before with the sum of two die. Since each of Alice and Bob rolls one 6-sided die, then there are $6 \times 6 = 36$ possible combinations of rolls.
Each of these 36 possibilities is equally likely.
Alice wins when the two values rolled differ by 1. The possible combinations that differ by 1 are (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (2, 1), (3, 2), (4, 3), (5, 4), and (6, 5).
Therefore, there are 10 combinations when Alice wins.
Thus her probability of winning is $\frac{10}{36} = \frac{5}{18}$.

12. * What is the probability of hitting a bullseye on a dartboard if the bullseye has a radius of 1cm and the board has a radius of 10cm?

Hint: *Area of a Circle* $= \pi \times r^2$ *where* $\pi \approx 3.14$ *and* $r^2 = radius \times radius$

The probability of hitting a bullseye can be found as follows:

$$P(Hitting\ a\ bullseye) = \frac{Area\ of\ the\ bullseye}{Total\ available\ area\ on\ the\ board}$$

Using the hint, the area of the bullseye $A_{bullseye} = \pi \times 1cm \times 1cm = \pi cm^2$ and the area of the board $A_{board} = \pi \times 10cm \times 10cm = 100\pi cm^2$.

Then the probability of hitting a bullseye is:

$$P(Hitting\ a\ bullseye) = \frac{\pi\ cm^2}{100\ \pi\ cm^2} = \frac{1}{100} = 0.01 = 1\%$$

13. * In Canada, 13% of the population plays hockey, basketball and baseball. Additionally, 25% of the population plays basketball and hockey, 16% plays basketball and baseball and 21% plays hockey and baseball. If 28% of the population only play basketball and 15% play only baseball, what percent of the population plays hockey?

Hint: *Use a Venn Diagram to help you visualize.*

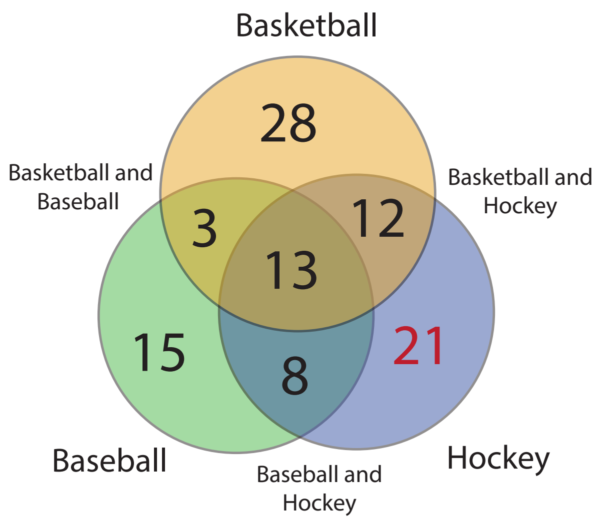
We use a 3 circle diagram such as the one pictured below to complete this question.

We are given that 13% of the population plays all 3 sports so we can fill the intersection of all 3 circles with 13. We are then given that 25% of the population plays basketball and hockey but we must remember that 13% of those are the ones that play all 3 sports so only 12% play only basketball and hockey and so we fill the intersection of of the basketball circle and hockey circle with 12. In a similar way we find 3 and 8. Lastly, 28% of the population plays only basketball and no other sports so we fill the remaining part of the basketball circle with 28 and we do the same for the baseball circle to get 15. Now we have all numbers except one but we know that the entire population plays at least one sport so the total of these numbers must add up to 100%. So 21% of the population plays only hockey since:

P(Play **only** hockey) = 100% - 28% - 3% - 13% - 12% - 15% - 8% = 21%

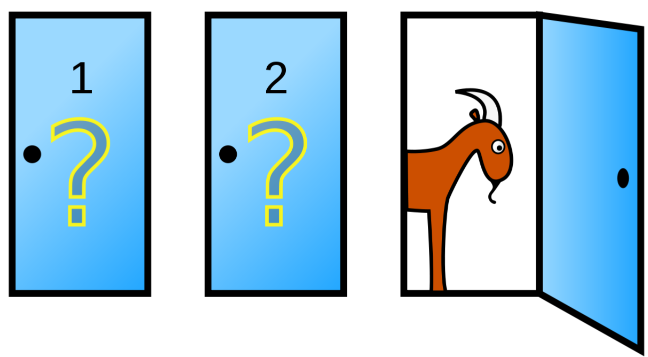
But we want the percent of the population that plays hockey which is:

P(Play hockey) = 12% + 13% + 8% + 21% = 54%.



14. * **The Monty Hall Problem**

This is a famous math problem that deals with probability.



You are on a gameshow where you're asked to pick one of three closed doors. Behind two of the three doors there are goats. But behind one of them, there's a brand new car.

(a) What is the probability of winning the car?

You want to find the probability of winning the car or P(car). You know that there are 3 possible outcomes in this experiment of picking a door at random.

Since youve got 2 goats and 1 car, theres only 1 way of picking the door with the car behind it. This means that:

$$P(car) = \frac{Number\ of\ ways\ of\ picking\ the\ car}{Total\ number\ of\ possible\ outcomes} = \frac{1}{3}$$

So the probability of picking the car is $\frac{1}{3}$.

(b) You've now picked a door. The gameshow host opens one of the doors you didn't pick and reveals a goat. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door. Should you change your mind and pick the other door? Why or why not?

You have now picked a door. The gameshow host opens one of the doors you didn't pick to reveal a goat. You now have two closed doors, one of which is the one you picked. You're given a chance to change the door you picked.

The question asks you whether or not you should change your choice.

To answer this question, we need to understand what happens to P(car) when the host opens one of the doors with a goat behind it.

A good number of people would say that it doesn't matter if you change your choice or not since $P(car)$ increases to $\frac{1}{2}$ or 50%. According to them, this is because the number of possible outcomes is now reduced to 2 instead of 3.

Although this answer sounds right, it's actually **wrong**.

The reason has to do with the fact that you made the choice when all three doors were closed (i.e. there were 3 possible outcomes).

When you first pick your door, $P(car) = \frac{1}{3}$. This means that the car has a $\frac{2}{3}$ probability of being behind one of the doors you **didn't** pick.

Now, when the host opens one of the doors you didn't pick and reveals a goat, then there's a $\frac{2}{3}$ probability that the car is behind the other closed door.

Since there's a 2 in 3 chance that the car is behind the remaining closed door (that you didn't pick), the answer is that **you should change your choice every single time**.

To see how this solution works, it might be helpful to think of a game with 100 doors, 99 goats and 1 car. As before, you pick a door at random. There's a $\frac{1}{100}$ or 1% chance that the car is behind the door you picked.

If the host opens 98 of the remaining doors and reveals goats behind each and every one of them, then there's a $\frac{99}{100}$ chance that the car is behind the last closed door that you didn't pick. So if you're given a chance to change your choice, you should in order to increase your chances of winning.

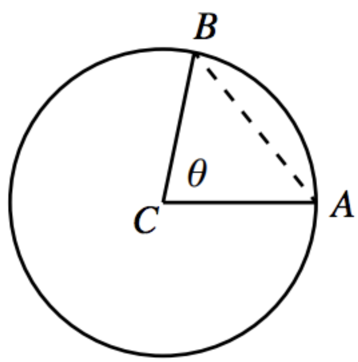
15. * Three friends are in the park. Bob and Clarise are standing at the same spot and Abe is standing 10 m away. Bob chooses a random direction and walks in this direction until he is 10 m from Clarise. What is the probability that Bob is closer to Abe than Clarise is to Abe? (Source: 2014 Cayley (Grade 10), #23)

We call Clarises spot C and Abes spot A.

Consider a circle centred at C with radius 10 m. Since A is 10 m from C, then A is on this circle.

Bob starts at C and picks a direction to walk, with every direction being equally likely to be chosen. We model this by having Bob choose an angle θ between $0°$ and $360°$ and walk 10 m along a segment that makes this angle when measured counterclockwise from CA.

Bob ends at point B, which is also on the circle.



We need to determine the probability that AB < AC.

Since the circle is symmetric above and below the diameter implied by CA, we can assume that θ is between $0°$ and $180°$ as the probability will be the same below the diameter.

Consider $\triangle CAB$ and note that $CA = CB = 10$ m.

It will be true that $AB < AC$ whenever AB is the shortest side of $\triangle ABC$.

AB will be the shortest side of $\triangle ABC$ whenever it is opposite the smallest angle of $\triangle ABC$.(In any triangle, the shortest side is opposite the smallest angle and the longest side is oppositethe largest angle.)

Since $\triangle ABC$ is isosceles with $CA = CB$, then $\angle CAB = \angle CBA$.

We know that $\theta = \angle ACB$ is opposite AB and $\angle ACB + \angle CAB + \angle CBA = 180^\circ$

Since $\angle CAB = \angle CBA$, then $\angle ACB + 2 \triangle CAB = 180^\circ$ or $\angle CAB = 90^\circ - \frac{1}{2}\angle ACB$.

If $\theta = \angle ACB$ is smaller than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be greater than 60°.

Similarly, if $\angle ACB$ is greater than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be smaller than 60°.

Therefore, AB is the shortest side of $\triangle ABC$ whenever θ is between 0° and 60°.

Since θ is uniformly chosen in the range 0° to 180° and $60^\circ = \frac{1}{3} \times 180^\circ$, then the probability that θ is in the desired range is $\frac{1}{3}$.

Therefore, the probability that Bob is closer to Abe than Clarise is to Abe is $\frac{1}{3}$

(Note that we can ignore the cases $\theta = 0^\circ$, $\theta = 60^\circ$ and $\theta = 180^\circ$ because these are only three specific cases out of an infinite number of possible values for θ.)