Energy Balance in Climate

With all the discussion of rapid climate change caused by man it may surprise you that in the past, the Earth's climate was much more variable than it is today. The reasons for this are a complex dance of astrophysics (for example the strength of Solar forcing, or the changes in the shape of the Earth's orbit) and geochemistry (for example the strength of the greenhouse effect due to gases in the Earth's atmosphere). At one point during the Neoproterozoic period some theories suggest nearly the entire globe was covered by ice, and this is called the Snowball Earth hypothesis.

A very simple model can be analyzed in terms of elementary differential equations and falls under the category of Energy Balance Models. We start by assuming the entire planet can be characterized by a single variable, namely its temperature. In words the model is written as

Change in the Total Heat of the Planet = Heat coming in from Sun − Heat radiated

or in equations

$$C\frac{dT}{dt} = S\left[1-\alpha(T)\right] + \epsilon\sigma T^4.$$

The dependent variable here is temperature (measured in Kelvin) and the independent variable is time. As with many physical theories there are many constants involved, most of which we will happily set to 1 soon enough. For completeness, C is the heat capacity, S is the solar insolation, σ is the Stefan-Boltzmann constant and ϵ is the permittivity. The one interesting physical law here that you may not have seen before is what is called the Stefan-Boltzmann law for the energy radiated by a perfect, or black, body: $E = \sigma T^4$. This says that hotter objects radiate much more energy, and if you visit a black smith's shop you will see a manifestation of this in how the heated metal glows. In any event, the property of our planet we want to vary is how much energy it reflects or absorbs. This is called *albedo* and is represented by the parameter $\alpha(T)$. There are many detailed studies of albedo, but the one you may recall from grade school is that dark objects absorb much more heat than white objects. This means that ice reflects, while ice free regions absorb heat.

The left hand side is calculus for "rate of change" and we won't end up using it a whole lot for these problems.

Question 1 Let's get some intuition for the black body law, often named Stefan's law.
i) Calculate the increase of energy radiated when the temperature is increase by one degree starting at 0, 20, 100 and $6,000$ degrees. Remember to add 273 to each of these when plugging into Stefan's law to change the temperature units to Kelvin. The four values are: the freezing point of water, room temperature, the boiling point of water and the approximate temperature of our Sun.

ii) Use the values you found in i) to calculate the relative change.

iii) Consider Stefan's law written with a change in temperature like this

$$E = \sigma(T + \Delta T)^4.$$

Let

$$E_0 = \sigma T^4$$

and consider the relative change

$$\frac{E - E_0}{E_0} = \frac{(T + \Delta T)^4 - T^4}{T^4}.$$

Now use the binomial theorem and the fact that

$$\frac{\Delta T}{T} \ll 1$$

to find a simple formula for the relative change in terms of T and ΔT.

iv) Use your new formula to compare to the values you found in ii).

OK now that you have a bit of intuition for Stefan's Law, let's go back to the really simple climate model and do something with it. We want to consider the equation above, but not all of us have seen Calculus yet. So let's do something that applied mathematicians do anyway; let's say we only want to find the possible states of the planet for which the temperature doesn't change. This is the same as saying that the rate of change with time equals zero, or that the left hand side (which conveniently has all the calculus) is zero.

Question 2 The two albedo model. Let's assume the albedo has two possible values so that

$$\alpha(T) = \begin{cases} 0.7 & T \leq 273 \\ 0.2 & T > 273 \end{cases}$$

i) Sketch the right hand side of the equation and show that it has a jump, or discontinuity, at $T = 0$ Centigrade and has precisely two values of T for which it equals zero.

ii) Choosing parameters $\sigma = 5.6 \times 10^{-8}$, $\epsilon = 0.7$ and $S = 400$ find the two possible “equilibrium” temperature values of the climate. You should find one below zero Centigrade (273 Kelvin) and one above zero Centigrade. The cold one is, somewhat ominously, called the Snowball Earth.

iii) But wait you say, how do we know this Snowball Earth could ever happen? Let’s recall what the left had side of the equation meant in words, namely the rate of change of temperature with time. That means the right hand side of equation tells us whether temperature will grow (positive rate of change) or decrease (negative rate of change). Analyze what happens near the two equilibrium points you found in part ii. You should find that if the temperature is “nudged” a little bit, both the warm and cold states of the planet are pushed back to the equilibrium. This is called a “stable” equilibrium (I can show you an example with a stick).

Together parts ii) and iii) suggest it is possible to have a stable Snowball Earth climate! And in fact there is evidence both from fossils, and climate models that something like a Snowball Earth happened in the distant past.

Researchers often challenge simple models almost right away and indeed the idea of an instant transition between white ice and dark, ice–free ground at $T = 0$ seems a bit silly. A more complex model is given by

$$\alpha(T) = \begin{cases} \alpha_1 & T < T_1 \\ \alpha_1 + (\alpha_2 - \alpha_1)\frac{T-T_1}{T_2-T_1} & T_1 < T < T_2 \\ \alpha_2 & T_2 < T \end{cases}$$

and for concreteness let's choose $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $T_1 = 263$ and $T_2 = 273$. This means the in between region is in the range minus ten to ten degrees Celsius, and that seems like a reasonable place to start.

Question 3

i) Sketch the albedo and show that the albedo is now continuous but that it has a sharp change of slope (not differentiable in calculus language).

ii) Choose the same parameters as in Question 1 and show that there are three equilibrium points.

iii) Analyze the stability of each of the three equilibrium points (graphically) and interpret your findings. You should find that once again the hot and frozen states are stable but that there is a mushy state in which the temperature is near zero Centigrade (273 Kelvin).